Modeling Drift

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Contents

1 Introduction **1**

2 Brownian Motion notes **2**
2.1 Estimating the diffusion constant . . . 2
2.2 Limits of Power . . . 2
2.3 Changing D . . . 3
2.4 Peak Values . . . 5
2.5 Peak after applying temporal filter . . . 8

3 Appendix **8**
3.1 Diffusion Equation & Solutions . . . 8

4 Window Type, Smoothing **11**

5 Comparison of Different Sampling Frequencies **13**

6 Compairson of Different Window Sizes **15**

7 Too Short Windows **17**

1 Introduction

Here I test the quality of the QRad Welch Single Trace algorithm on randomly generated Brownian motion traces. This came up after theoretical and QRad power spectra for the Eye Chart Experiment did not match well.

Power spectra are estimated for given eye traces $x(t)$, $y(t)$ using Welch's method on the signal

$$e^{-2\pi i(\xi_x \cos(\theta)x(t)+\xi_y \sin(\theta)y(t))}$$

where ξ_x and ξ_y are spatial frequencies, θ is the angle in the spectrum which is averaged over to collapse the spatial dimensions.

2 Brownian Motion notes

The probability of gaze location varies with time as

$$\frac{\partial q}{\partial t} = D\frac{\partial^2 q}{\partial x^2} + D\frac{\partial^2 q}{\partial y^2}$$

or equivalently, when $r^2 = x^2 + y^2$ (see appendix for derivation)

$$\frac{\partial q}{\partial t} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial q}{\partial r}\right)$$

Comparisons are made with the theoretical power spectrum based on Brownian motion drifts:

$$q(x, y, t; D_R) = \frac{1}{4\pi Dt}\exp\left(-\frac{x^2+y^2}{4Dt}\right)$$
$$Q(\xi_x, \xi_y, f; D_R) = \frac{2D_R(\xi_x^2+\xi_y^2)}{4\pi^2 D_R^2(\xi_x^2+\xi_y^2)+f^2}$$
$$Q(\xi, f; D_R) = \frac{2D_R\xi^2}{4\pi^2 D_R^2\xi^4+f^2}$$

Note that this is often presented in angular frequencies ($k = 2\pi\xi, \omega = 2\pi f$) instead of ordinary frequencies (ξ, f):

$$Q(k, \omega; D) = \frac{2Dk^2}{D^2k^4+\omega^2}$$

2.1 Estimating the diffusion constant

For a 2-D isotropic, random walk, we know that

$$\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle$$

where $\langle x^2\rangle = \langle y^2\rangle = 2Dt$. Therefore,

$$\langle r^2\rangle = 4Dt \Rightarrow D = \frac{\langle r^2\rangle}{4t}$$

2.2 Limits of Power

Note that when $D_R = 0$ there is 0 power at $f \neq 0$. When $D_R = 0$ and $f = 0$, there is infinite power at all spatial frequencies.

Furthermore, note that when integrating over temporal frequencies to a limit (ideal LPF) the total temporal power is constant and invariant to spatial frequency and diffusion constant.

$$\int_0^L Q(\xi, f; D_R)\, df = \left[\frac{1}{\pi}\tan^{-1}\left(\frac{f}{2\pi D_R \xi^2}\right)\right]_0^L$$
$$= \frac{1}{\pi}\tan^{-1}\left(\frac{L}{2\pi D_R \xi^2}\right) \qquad \text{(log-log linear)}$$
$$\lim_{L\to\infty}\int_0^L Q(\xi, f; D_R)\, df = \frac{1}{2} \qquad \text{(constant)}$$

When integrating over a temporal frequency range (ideal bandpass):

$$\frac{1}{L_2 - L_1}\int_{L_1}^{L_2} Q(\xi, f; D_R)\, df = \frac{1}{\pi(L_2 - L_1)}\left(\tan^{-1}\left(\frac{L_2}{2\pi D_R \xi^2}\right) - \tan^{-1}\left(\frac{L_1}{2\pi D_R \xi^2}\right)\right)$$

The shape of this function:

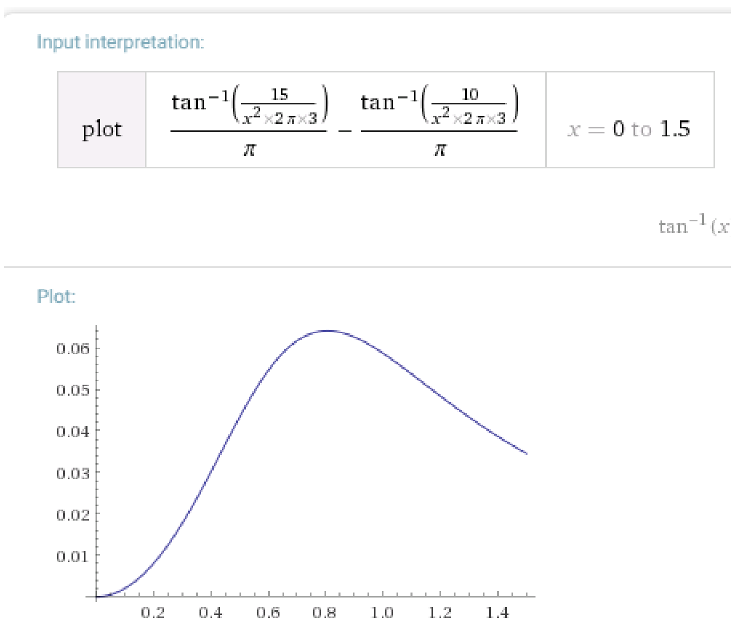


Figure 1: example shape of power vs spatial frequency from ideal bandpass

2.3 Changing D

Note that for two different retinal diffusion coefficients $D_2 = aD_1$, the resulting spatiotemporal power spectra are spatial frequency-scaled versions of one another:

$$Q(\xi, f; D_2) = Q(\xi, f; aD_1)$$
$$= \frac{2D_1 a\xi^2}{4\pi^2 D_a^2 a^2 \xi^4 + f^2}$$
$$= \frac{2D_1(\sqrt{a}\xi)^2}{4\pi^2 D_a^2 (\sqrt{a}\xi)^4 + f^2}$$
$$= Q(\sqrt{a}\xi, f; D_1)$$

This holds for each individual temporal frequency (see Fig 2) and even after temporal filtering:

$$\begin{aligned} S(\xi; D_2) &= \int_0^\infty H(f) Q(\xi, f; D_2)\, df \\ &= \int_0^\infty H(f) Q(\sqrt{a}\xi, f; D_1)\, df \\ &= S(\sqrt{a}\xi; D_1) \end{aligned}$$

where $H(f)$ is the frequency response of some low or bandpass filter. Therefore, the maximum value of $S(\xi; D_2)$ and $S(\xi; D_1) = S(\sqrt{a}\xi; D_1)$ are the same, though the peaks occur at different spatial frequencies for $a \neq 1$.

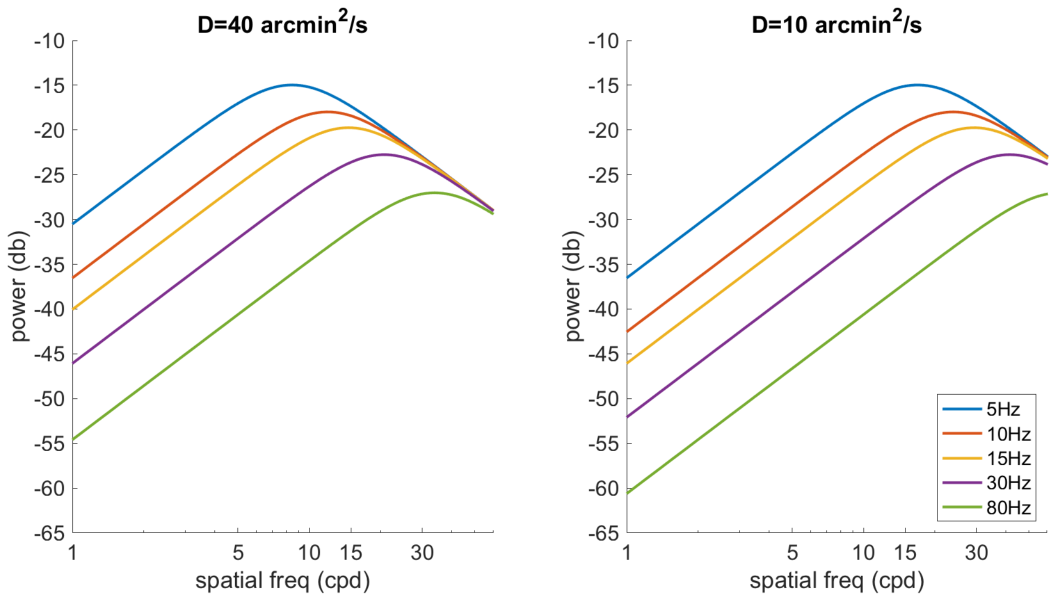


Figure 2: Power in different temporal frequency bands for two diffusion coefficients. Note that changing the diffusion coefficient scales the x-axis so that the the peak values is the same regardless of the diffusion coefficient.

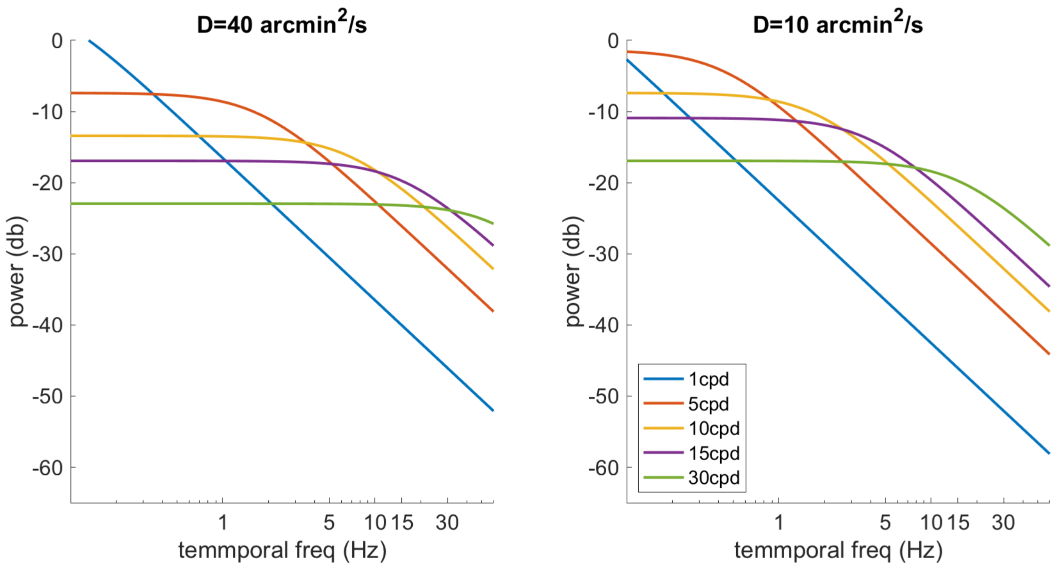


Figure 3: Power in different spatial frequency bands for two diffusion coefficients.

2.4 Peak Values

Here we compute the spatial frequency at which peak power occurs for a given (f) and show that the value of this peak does not depend on the diffusion constant D_R.

Given D_R and f:

$$\begin{aligned}\frac{\partial Q}{\partial \xi} = 0 = & \frac{(2^2 D_R \xi)(2^2\pi^2 D_R^2 \xi^4 + f^2) - (2D_R\xi^2)(2^4\pi^2 D_R^2 \xi^3)}{(2^2\pi^2 D_R^2\xi^4 + f^2)^2} \\ = & \frac{(2D_R\xi)(-2^3\pi^2 D_R^2\xi^4 + 2f^2)}{(2^2\pi^2 D_R^2\xi^4 + f^2)^2} \\ \Rightarrow & 2D_R\xi = 0 \rightarrow \xi = 0 \quad \text{(minimum)} \\ \Rightarrow & 2^3\pi^2 D_R^2 \xi'^4 + 2f^2 = 0 \rightarrow \boxed{\xi'^2 = \frac{f}{2\pi D_R}}\end{aligned}$$

Therefore, we know that power peaks at the spatial frequency $\xi' = \sqrt{f/2\pi D}$ as shown in Figure 4.

Note that when $D_R = 0$, the power peaks at an infinite spatial frequency so that the power in any temporal frequency band cannot vary and must remain constant at 0 - except at $f = 0$ where power is infinite and constant

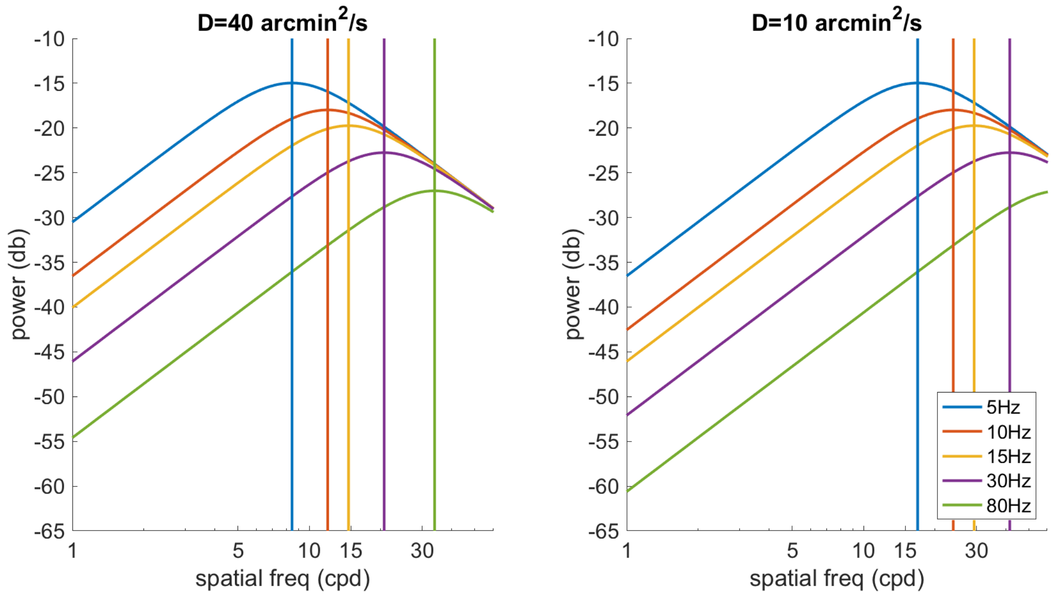


Figure 4: Power in different frequency bands for two diffusion coefficients. Note that changing the diffusion coefficient scales the x-axis so that the the peak values is the same regardless of the diffusion coefficient. The peak location can be predicted as described above.

Given the spatial frequency ξ' at which the peak occurs, we can compute the value of power at this point:

$$\begin{aligned} Q(\xi', f; D_R) &= \frac{2D_R \frac{f}{2\pi D}}{4\pi^2 D_R^2 \frac{f^2}{2^2\pi^2 D_R^2} + f^2} \\ &= \boxed{\frac{1}{2\pi f}} \end{aligned}$$

Therefore, the peak power in each temporal frequency band is determined only by the temporal frequency (see Figure 5)- and is in fact proportional to the inverse of the temporal frequency.

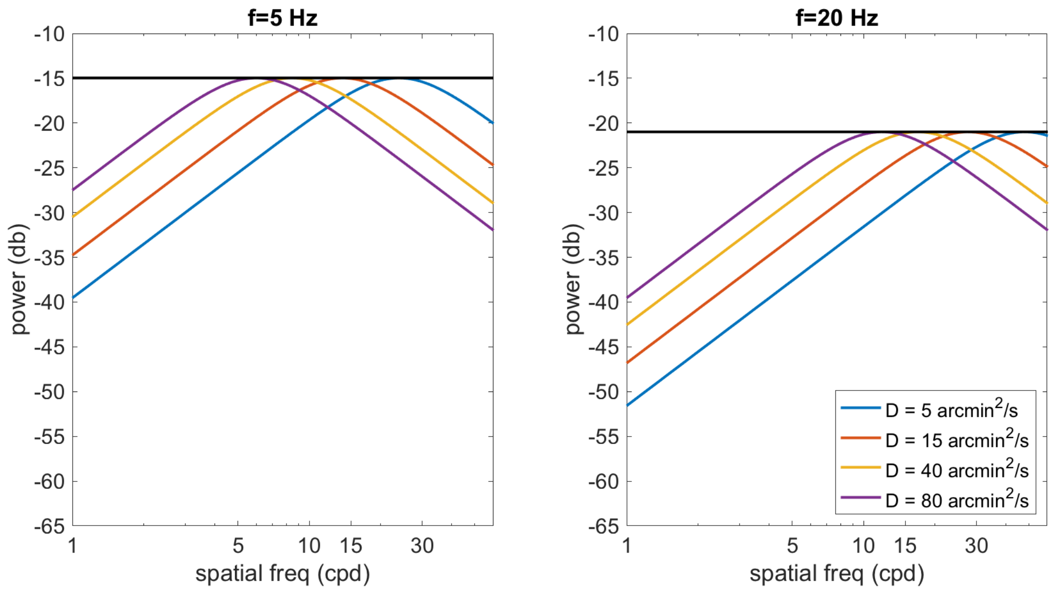


Figure 5: Power for different diffusion constants in two temporal frequency bands. Changing the diffusion constant varies the location of the peak, but the peak value is invariant to the diffusion coefficient.

In log-log-scale, the "stretching" of the spatial frequency axis may appear to be more like a shift. The stretching is more evident when axes are shown in linear scales as shown in Figure 6.

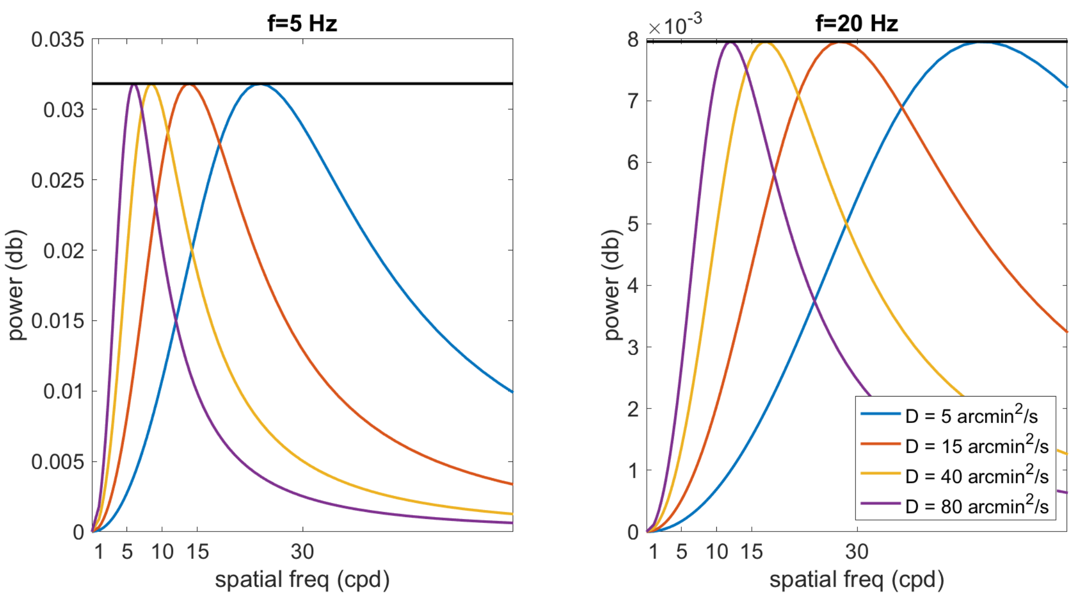


Figure 6: Same as 5 except in linear scale to show stretching of spatial frequency better.

2.5 Peak after applying temporal filter

If a temporal filter $H(f)$ is applied, then the power spectrum peaks when: (analagous to above)

$$\begin{aligned}
\int H(f)\frac{\partial Q}{\partial \xi}\,df = 0 &= \int \frac{(2^2 D_R\xi)(2^2\pi^2 D_R^2\xi^4 + f^2) - (2D_R\xi^2)(2^4\pi^2 D_R^2\xi^3)}{(2^2\pi^2 D_R^2\xi^4 + f^2)^2} H(f)\,df \\
&= \int \frac{(2D_R\xi)(-2^3\pi^2 D_R^2\xi^4 + 2f^2)}{(2^2\pi^2 D_R^2\xi^4 + f^2)^2} H(f)\,df \\
&\Rightarrow 2D_R\xi = 0 \rightarrow \xi = 0 \quad \text{(minimum)} \\
&\Rightarrow 2^3\pi^2 D_R^2\xi'^4 \int H(f)\,df + 2\int f^2 H(f)\,df = 0 \rightarrow \boxed{\xi'^4 = \frac{\int f^2 H(f)\,df}{2^2\pi^2 D_R^2 \int H(f)\,df}}
\end{aligned}$$

Letting $\int H(f)\,df = 1$, then the integral in the numerator is the second moment of $H(f)$, $h_2 = \int f^2 H(f)\,df$.
Then, the peak value is

$$\begin{aligned}
Q(\xi', f; D_R) &= \frac{2D_R\frac{\sqrt{h_2}}{2\pi D_R}}{4\pi^2 D_R^2 \frac{h_2}{2^2\pi^2 D_R^2} + f^2} \\
&= \boxed{\int \frac{\sqrt{h_2}}{\pi(h_2 + f^2)} H(f)\,df}
\end{aligned}$$

3 Appendix

3.1 Diffusion Equation & Solutions

A 2-dimensional random walk follows the diffusion equation

$$\frac{\partial q}{\partial t} = D\frac{\partial^2 q}{\partial x^2} + D\frac{\partial^2 q}{\partial y^2}$$

where D is the diffusion constant on each independent dimension. We can also write this as a function of $r^2 = x^2 + y^2$:

$$r = \left(x^2+y^2\right)^{1/2}$$
$$\frac{\partial r}{\partial x} = \frac{x}{r}$$
$$\frac{\partial^2 r}{\partial x^2} = \frac{r^2-x^2}{r^3} = \frac{y^2}{r^3}$$

$$\begin{aligned}
\frac{\partial q}{\partial t} &= D\left[\frac{\partial}{\partial x}\left(\frac{\partial q}{\partial r}\cdot\frac{\partial r}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial q}{\partial r}\cdot\frac{\partial r}{\partial y}\right)\right] \\
&= D\left[\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial q}{\partial r}\right)}_{\frac{\partial^2 q}{\partial r^2}\cdot\frac{\partial r}{\partial x}}\cdot\frac{\partial r}{\partial x}+\frac{\partial q}{\partial r}\frac{\partial^2 r}{\partial x^2}+\underbrace{\frac{\partial}{\partial y}\left(\frac{\partial q}{\partial r}\right)}_{\frac{\partial^2 q}{\partial r^2}\cdot\frac{\partial r}{\partial y}}\cdot\frac{\partial r}{\partial y}+\frac{\partial q}{\partial r}\frac{\partial^2 r}{\partial y^2}\right] \\
&= D\left[\frac{\partial^2 q}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2+\frac{\partial^2 q}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2+\frac{\partial q}{\partial r}\left(\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}\right)\right] \\
&= D\left[\frac{\partial^2 q}{\partial r^2}\left(\frac{x^2+y^2}{r^2}\right)+\frac{\partial q}{\partial r}\left(\frac{y^2+x^2}{r^3}\right)\right] \\
&= D\left[\frac{\partial^2 q}{\partial r^2}+\frac{1}{r}\frac{\partial q}{\partial r}\right] = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q}{\partial r}\right)
\end{aligned}$$

With the initial condition that at $t=0$ the particle is concentrated at 0, the diffusion equation is satisfied by

$$\begin{aligned}
q(x,y,t;D) &= \frac{1}{4\pi Dt}\exp\left(-\frac{x^2+y^2}{4Dt}\right) \\
q(r,t;D) &= \frac{1}{4\pi Dt}\exp\left(-\frac{r^2}{4Dt}\right)
\end{aligned}$$

Here we confirm that this solves the diffusion equation:

$$\begin{aligned}
\frac{\partial q}{\partial x} &= -\frac{2x}{4Dt}q \\
\frac{\partial^2 q}{\partial x^2} &= -\left(\frac{2}{4Dt}\right)q+\left(\frac{2x}{4Dt}\right)^2 q \\
&= \left(\frac{x^2-2Dt}{4D^2t^2}\right)q \\
\frac{\partial^2 q}{\partial x^2} &= \left(\frac{y^2-2Dt}{4D^2t^2}\right)q \\
\frac{\partial q}{\partial t} &= \left(-\frac{1}{t}+\frac{x^2+y^2}{4D^2t^2})\right)q \\
&= \frac{x^2+y^2-4Dt}{4Dt^2}q \\
\Rightarrow D\frac{\partial^2 q}{\partial x^2}+D\frac{\partial^2 q}{\partial y^2} &= \left(\frac{x^2-2Dt}{4Dt^2}\right)q+\left(\frac{x^2-2Dt}{4Dt^2}\right)q \\
&= \frac{x^2+y^2-4Dt}{4Dt^2}q = \frac{\partial q}{\partial t}
\end{aligned}$$

In three dimensions, the diffusion equation becomes:

$$\frac{\partial q}{\partial t} = D\frac{\partial^2 q}{\partial x^2} + D\frac{\partial^2 q}{\partial y^2} + D\frac{\partial^2 q}{\partial z^2}$$
$$\frac{\partial q}{\partial t} = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial q}{\partial r}\right)$$

4 Window Type, Smoothing

Parameters:

sampling frequency = 1000 Hz
10 trials of 10 seconds each
window length = 512 (square or hanning)
number of angles for radial average: 30
diffusion coefficient $D = 325.89$ arcmin^2/s

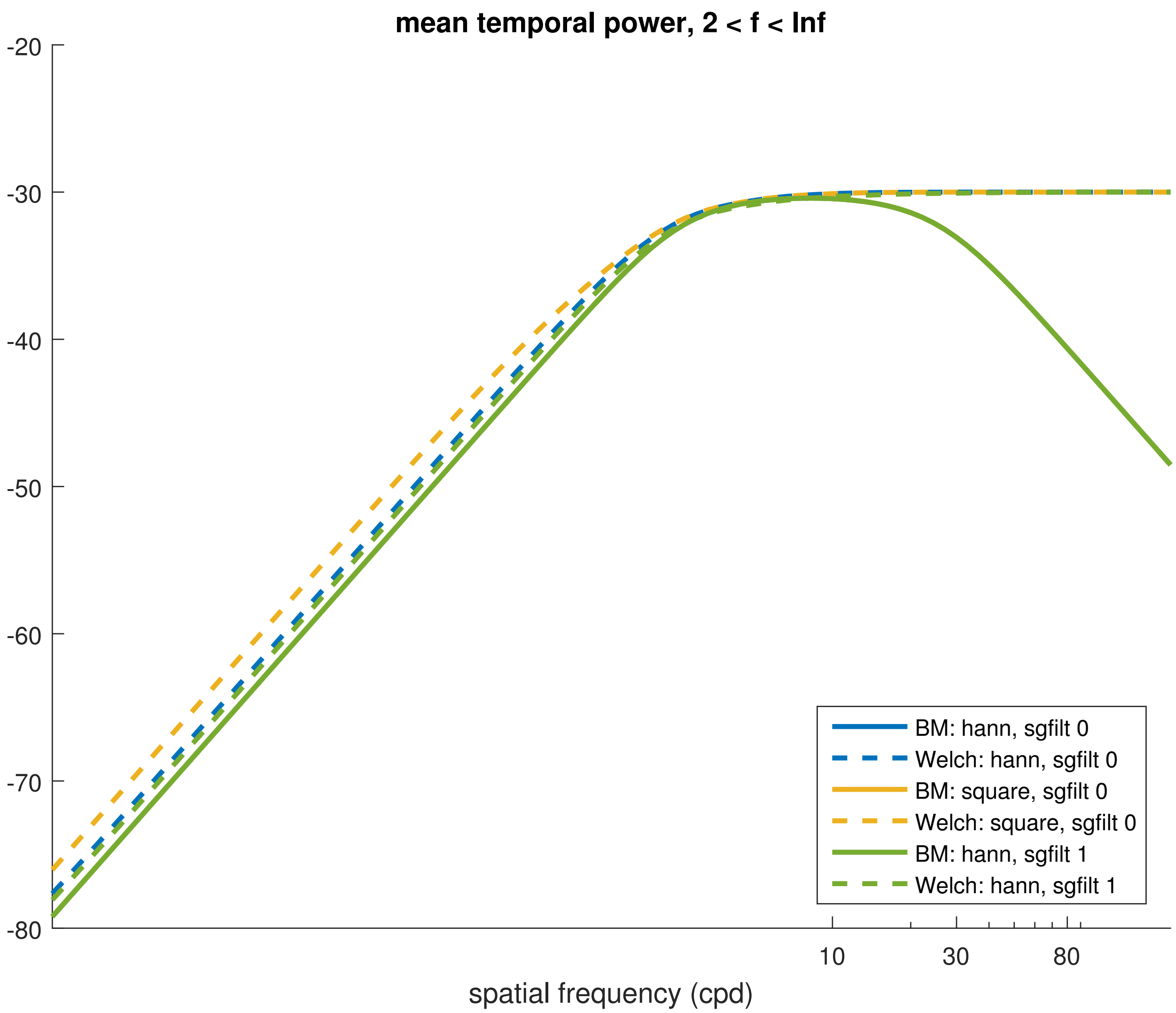
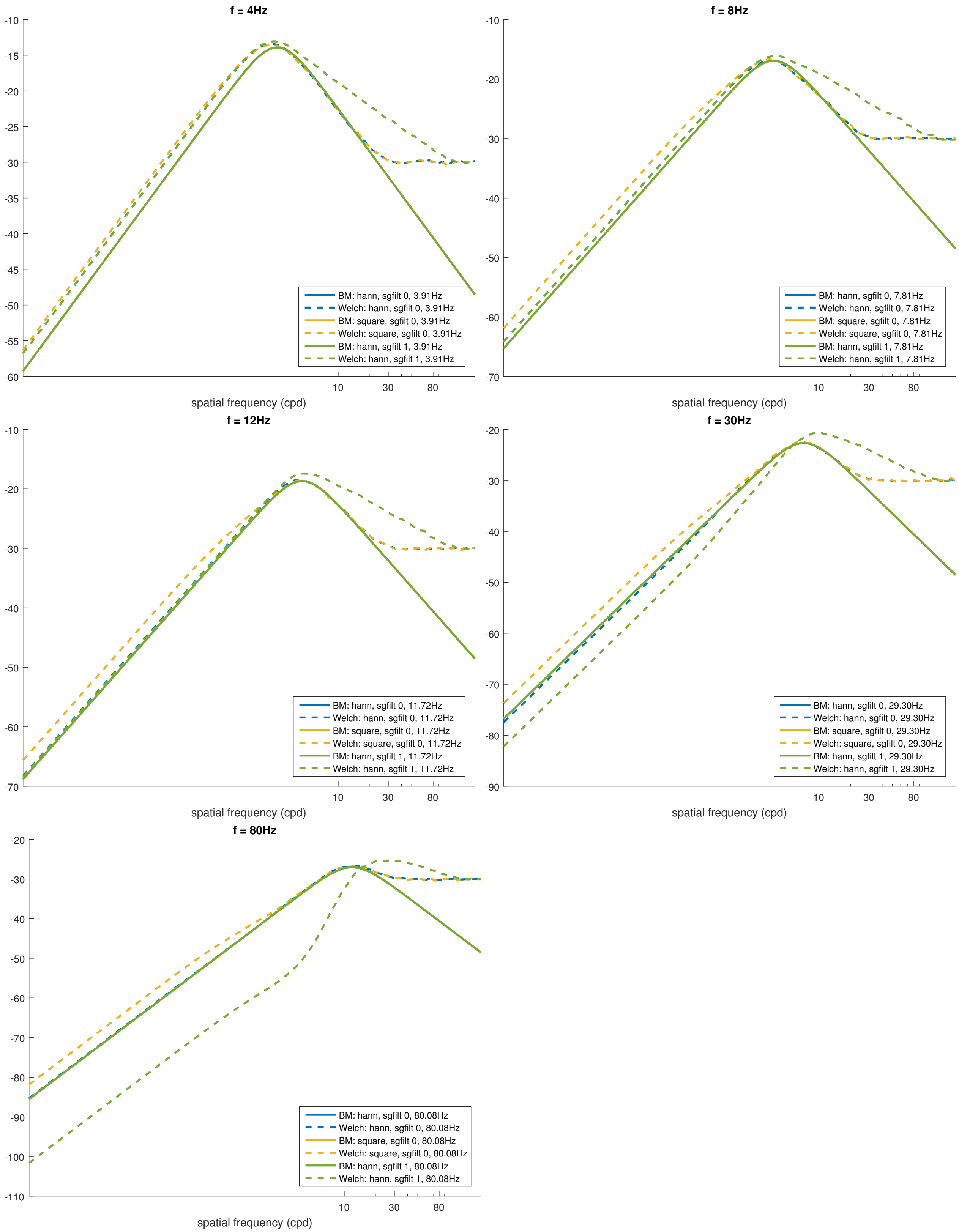


Figure 7: Mean temporal power up to Nyquist


f = 4Hz
-10
-15
-20
-25
-30
-35
-40
-45
-50
-55
-60
10
30
80
spatial frequency (cpd)
BM: hann, sgfilt 0, 3.91Hz
Welch: hann, sgfilt 0, 3.91Hz
BM: square, sgfilt 0, 3.91Hz
Welch: square, sgfilt 0, 3.91Hz
BM: hann, sgfilt 1, 3.91Hz
Welch: hann, sgfilt 1, 3.91Hz
f = 8Hz
-10
-20
-30
-40
-50
-60
-70
10
30
80
spatial frequency (cpd)
BM: hann, sgfilt 0, 7.81Hz
Welch: hann, sgfilt 0, 7.81Hz
BM: square, sgfilt 0, 7.81Hz
Welch: square, sgfilt 0, 7.81Hz
BM: hann, sgfilt 1, 7.81Hz
Welch: hann, sgfilt 1, 7.81Hz
f = 12Hz
-10
-20
-30
-40
-50
-60
-70
10
30
80
spatial frequency (cpd)
BM: hann, sgfilt 0, 11.72Hz
Welch: hann, sgfilt 0, 11.72Hz
BM: square, sgfilt 0, 11.72Hz
Welch: square, sgfilt 0, 11.72Hz
BM: hann, sgfilt 1, 11.72Hz
Welch: hann, sgfilt 1, 11.72Hz
f = 30Hz
-20
-30
-40
-50
-60
-70
-80
-90
10
30
80
spatial frequency (cpd)
BM: hann, sgfilt 0, 29.30Hz
Welch: hann, sgfilt 0, 29.30Hz
BM: square, sgfilt 0, 29.30Hz
Welch: square, sgfilt 0, 29.30Hz
BM: hann, sgfilt 1, 29.30Hz
Welch: hann, sgfilt 1, 29.30Hz
f = 80Hz
-20
-30
-40
-50
-60
-70
-80
-90
-100
-110
10
30
80
spatial frequency (cpd)
BM: hann, sgfilt 0, 80.08Hz
Welch: hann, sgfilt 0, 80.08Hz
BM: square, sgfilt 0, 80.08Hz
Welch: square, sgfilt 0, 80.08Hz
BM: hann, sgfilt 1, 80.08Hz
Welch: hann, sgfilt 1, 80.08Hz

5 Comparison of Different Sampling Frequencies

50 trials of 2 or 3s lengths at three different sampling frequencies (1000, 10000, 100000 Hz) $D = 300$ arcmin^2/s plus some noise so that the diffusion varied a bit from trial to trial.

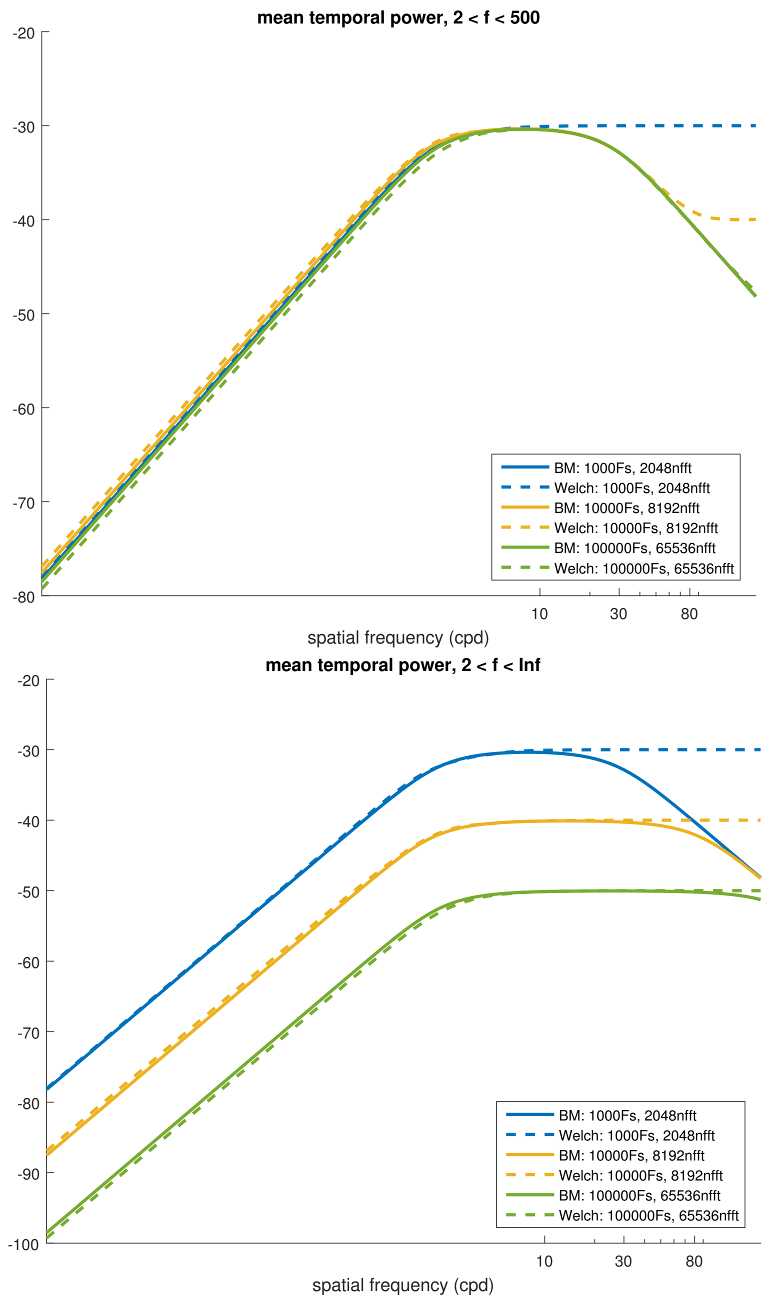


Figure 8: TOP: Mean of power up to 500Hz. BOTTOM: Mean of power up to Nyquist. Note that the saturation region extends as the sampling frequency increases.

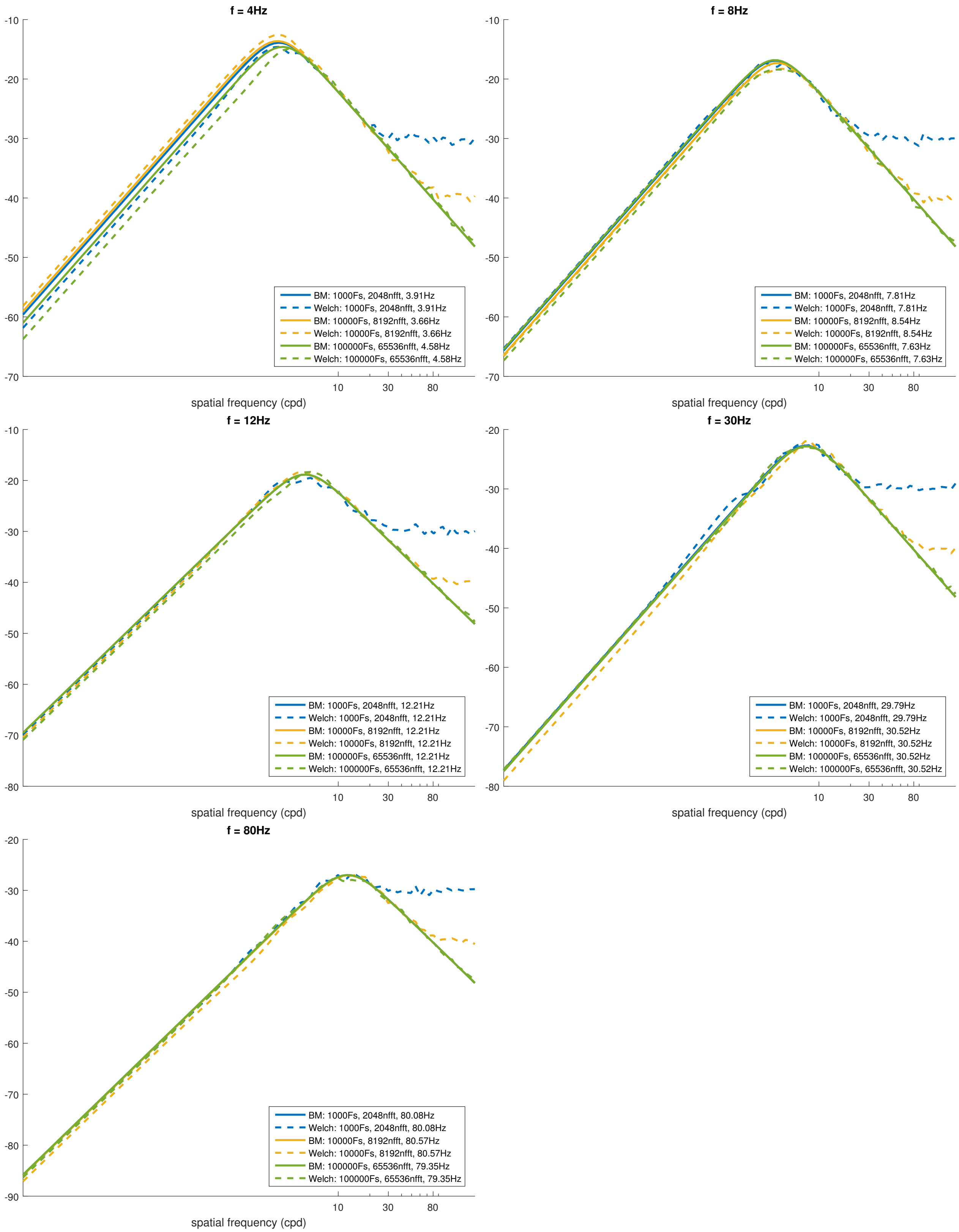


Figure 9: Slices of the power spectra at single temporal frequencies.

6 Compairson of Different Window Sizes

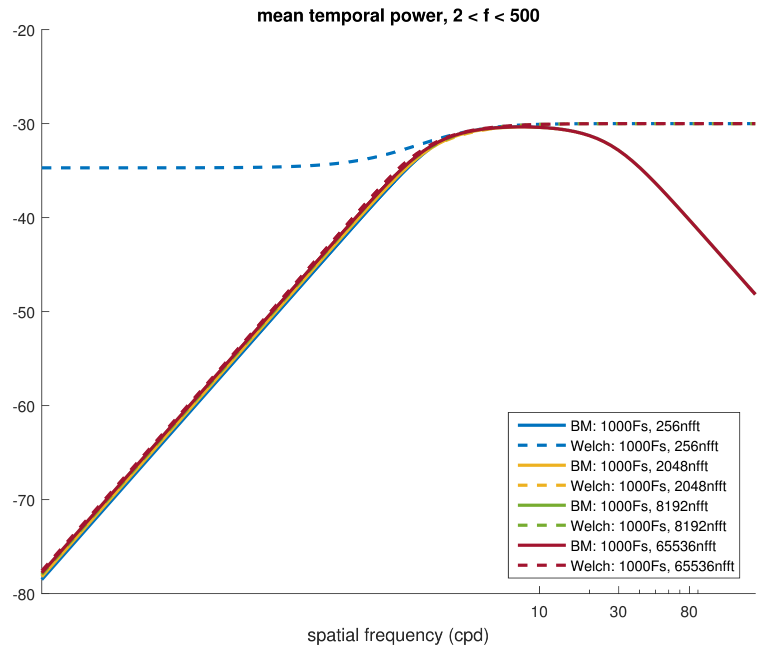


Figure 10

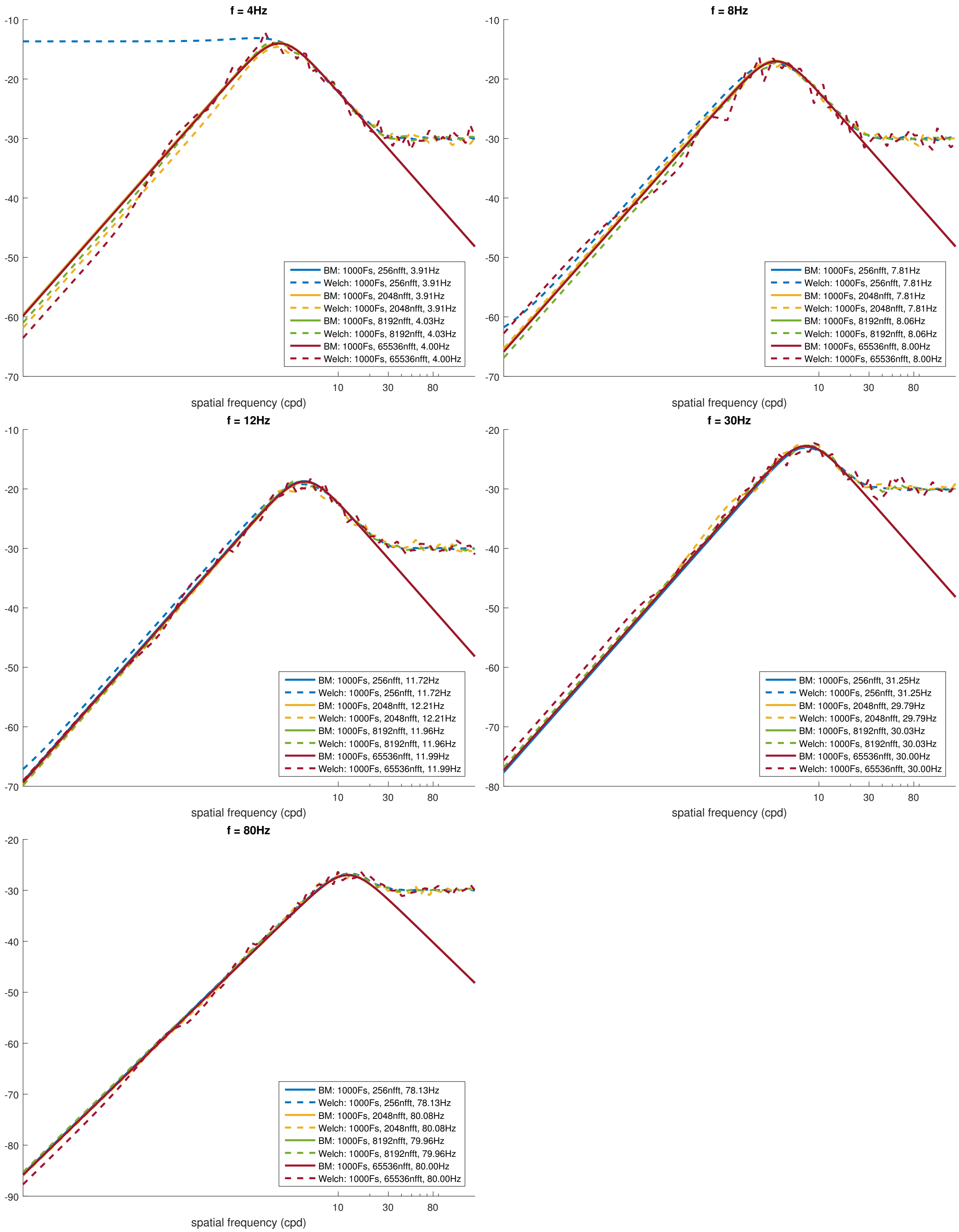


Figure 11: Slices of the power spectra at single temporal frequencies.

7 Too Short Windows

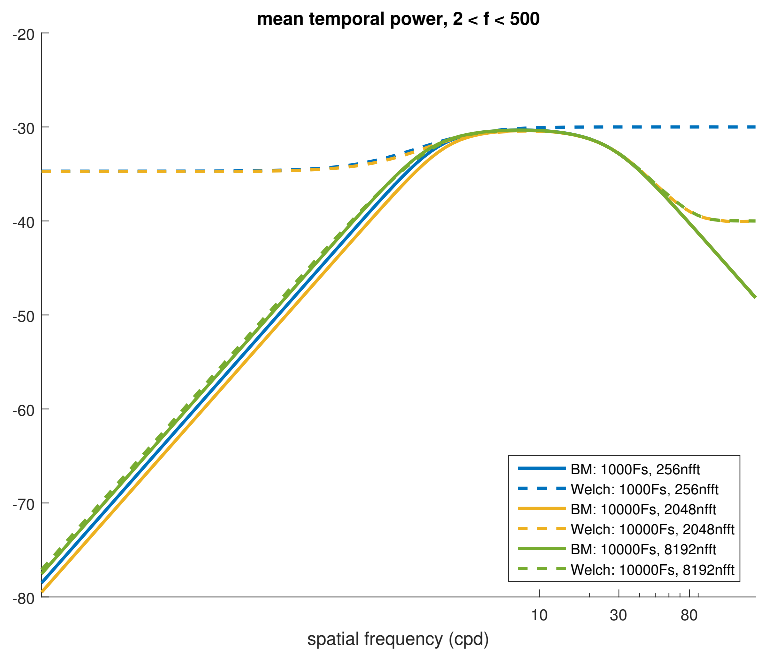


Figure 12

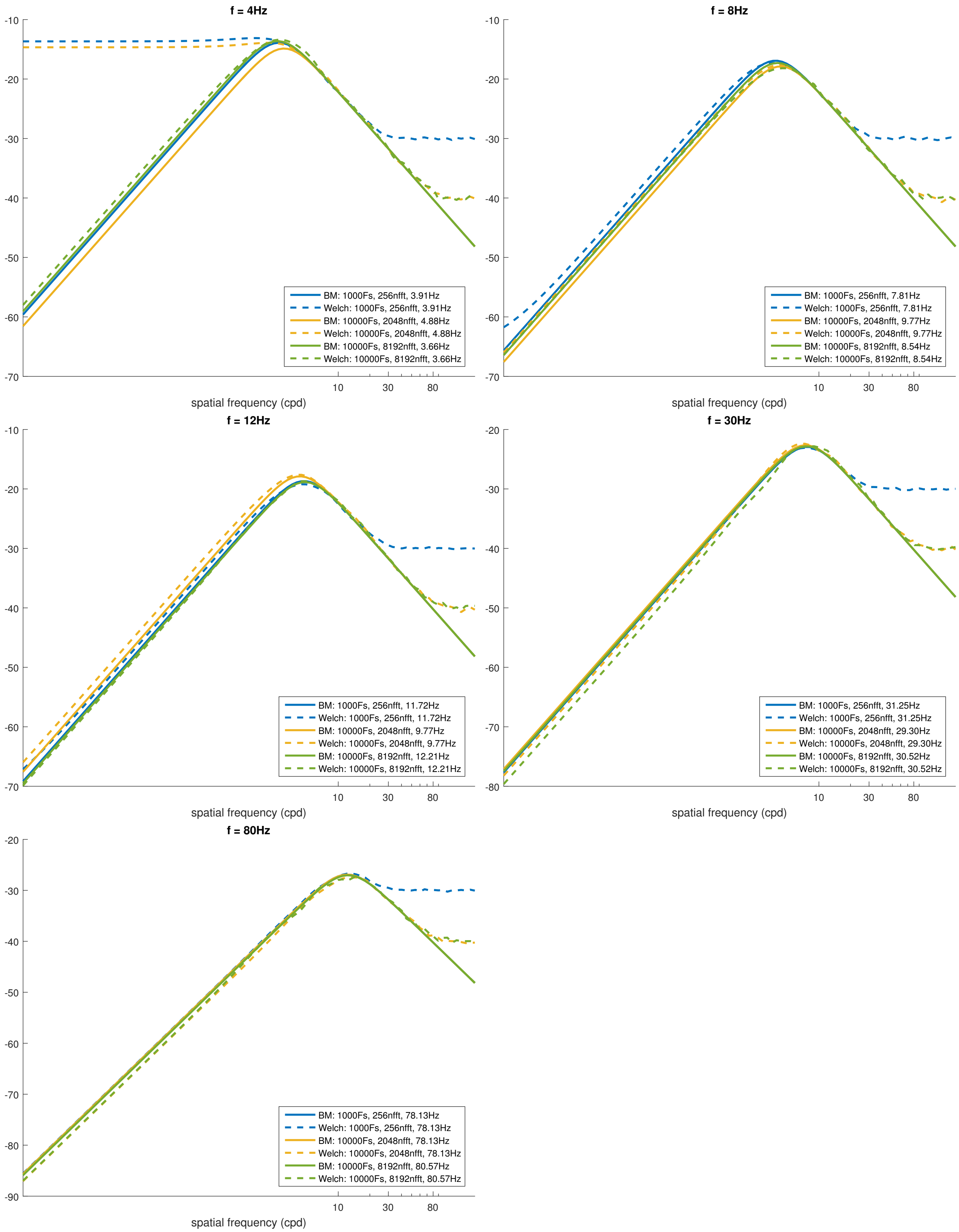


Figure 13: Slices of the power spectra at single temporal frequencies.