Estimation of spatiotemporal power spectrum in 1D

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1 Problem

Here we consider the estimation of power spectrum of retinal inputs under natural condition, i.e., a static input $I(\mathbf{x})$ under eye movements $\xi(t) = [\xi_x(t), \xi_y(t)]$. Our traditional way for this estimation requires to reconstruct a 3D movie of the retinal inputs, and then use the Welch method to calculate the power spectrum.

If we want to have higher resolution in spatial and temporal frequencies, we need to construct a larger 3D matrix for the 3D movie. This poses a challenge to the processing ability of the simulation software running under 32-bit operating system.

We want to find another way to estimate the power spectrum of retinal inputs.

2 Methods

2.1 Non-stationary eye movements

When a static image $L(\mathbf{x})$ ($\mathbf{x} = [x, y]$) is shifted under eye movements $\xi(t)$, the retinal input is given by $I(\mathbf{x}, t) = L[\mathbf{x} - \xi(t)]$. For a non-stationary eye movement process $\xi(t)$ (like a saccade), we are interested in

$$P_I(\mathbf{k}, \omega) = \left\langle |I(\mathbf{k}, \omega)|^2 \right\rangle_{\xi, L}, \tag{1}$$

where $\mathbf{k} = [k \cos \alpha, k \sin \alpha]$ is the spatial frequency, $I(\mathbf{k}, \omega)$ is the Fourier transform of $L[\mathbf{x} - \xi(t)]$,

$$I(\mathbf{k}, \omega) = \int dt e^{-2\pi i \omega t} \int d\mathbf{x} e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} L[\mathbf{x} - \xi(t)] \tag{2}$$

and $\langle\rangle_{\xi, L}$ represents an average over all eye traces ξ and all images L.

Substituting Eq. (2) into Eq. (1), we have

$$P_I(\mathbf{k}, \omega) = \left\langle \left| \int d\mathbf{x} e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} L(\mathbf{x}) \int dt e^{-2\pi i \omega t} e^{-2\pi i \mathbf{k} \cdot \xi(t)} \right|^2 \right\rangle_{\xi, L}. \tag{3}$$

Assuming eye movements are independent of the images, we can further simplify Eq. (3) as

$$P_I(\mathbf{k}, \omega) = P_L(\mathbf{k}) \left\langle \left| \int dt e^{-2\pi i \omega t} e^{-2\pi i \mathbf{k} \cdot \xi(t)} \right|^2 \right\rangle_{\xi} \tag{4}$$

where $P_L(\mathbf{k}) = \left\langle |L(\mathbf{k})|^2 \right\rangle_L$ is the power of the static input $L(\mathbf{x})$.

To reduce 2D space to 1D, we can conduct a radial average on $P_I(\mathbf{k}, \omega)$ over α. If $P_L(\mathbf{k})$ is radially symmetric, we have $P_L(k) = \langle P_L(\mathbf{k}) \rangle_\alpha$. Therefore, we obtain

$$P_I(k, \omega) = \langle P_I(\mathbf{k}, \omega) \rangle_\alpha = P_L(k) \left\langle \left| \int dt e^{-2\pi i \omega t} e^{-2\pi i \mathbf{k} \cdot \xi(t)} \right|^2 \right\rangle_{\xi, \alpha} \tag{5}$$

We can observe that the estimation of spatiotemporal power $P_I(k, \omega)$ can be conducted from a Fourier transform of $e^{-2\pi i \mathbf{k} \cdot \xi(t)}$.

This approach can also be applied when the static input is instead a spatiotemporal separable input, i.e., $I(\mathbf{x})W(t)$. In this case,

$$P_I(k, \omega) = P_L(k) \left\langle \left| \int dt e^{-2\pi i \omega t} W(t) e^{-2\pi i \mathbf{k} \cdot \xi(t)} \right|^2 \right\rangle_{\xi, \alpha}. \tag{6}$$

Now the power spectrum can be estimated by a temporal Fourier transform of $W(t)e^{-2\pi i \mathbf{k} \cdot \xi(t)}$.

2.2 Stationary eye movements

Eq. (1) is not the traditional way of defining power spectrum (without averaging over space and time), and should be used only for non-stationary eye movements. However, when $\xi(t)$ is a stationary process (i.e., drifts), we can consider the traditional way of power spectrum definition.

The finite fourier transform of $L[\mathbf{x} - \xi(t)]$ in time and space is given by

$$\begin{aligned} I_{\mathbf{W},T}(\mathbf{k}, \omega) &= \int_{-\mathbf{W}/2}^{\mathbf{W}/2} d\mathbf{x} \int_{-T/2}^{T/2} dt \; L[\mathbf{x} - \xi(t)] e^{-2\pi i (\mathbf{k} \cdot \mathbf{x} + \omega t)} \\ &= \int_{-T/2}^{T/2} dt \; e^{-2\pi i \omega t} e^{-2\pi i \mathbf{k} \cdot \xi(t)} L_{\mathbf{W}, \xi(t)}(\mathbf{k}), \end{aligned} \tag{7}$$

where

$$L_{\mathbf{W}, \xi(t)}(\mathbf{k}) = \int_{-\mathbf{W}/2 + \xi(t)}^{\mathbf{W}/2 + \xi(t)} d\mathbf{x}' \; L(\mathbf{x}') e^{-2\pi i \mathbf{k} \cdot \mathbf{x}'} \tag{8}$$

The power spectrum $P_L(\mathbf{k}, \omega)$ is therefore given by

$$\begin{aligned} P_I(\mathbf{k}, \omega) &= \lim_{\mathbf{W}, T \to \infty} \frac{1}{\mathbf{W}} \frac{1}{T} \left\langle |I_{\mathbf{W},T}(\mathbf{k}, \omega)|^2 \right\rangle_{\xi, L} \\ &= \lim_{\mathbf{W}, T \to \infty,} \frac{1}{\mathbf{W}} \frac{1}{T} \left\langle \left| \int_{-T/2}^{T/2} dt \; e^{-2\pi i \omega t} e^{-2\pi \mathbf{k} \cdot \xi(t)} L_{\mathbf{W}, \xi(t)}(\mathbf{k}) \right|^2 \right\rangle_{\xi, L}, \end{aligned} \tag{9}$$

and, assuming that eye movements are independent of the input $L(\mathbf{x})$, and $L_{\mathbf{W}, \xi(t)}(\mathbf{k}) \to L_{\mathbf{W}}(\mathbf{k})$ when $\mathbf{W} \to \infty$, we have

$$P_I(\mathbf{k}, \omega) = P_L(\mathbf{k}) P_E(\mathbf{k}, \omega), \tag{10}$$

where

$$P_L(\mathbf{k}) = \lim_{\mathbf{W}\to\infty} \frac{\left\langle |L_{\mathbf{W}}(\mathbf{k})|^2 \right\rangle_L}{\mathbf{W}} \tag{11}$$

is the power spectrum of the images, and

$$P_E(\mathbf{k}, \omega) = \lim_{T\to\infty} \frac{\left\langle \left| \int_{-T/2}^{T/2} dt\ e^{-2\pi i \omega t} e^{-2\pi i \mathbf{k}\cdot\xi(t)} \right|^2 \right\rangle_\xi}{T}, \tag{12}$$

denoting the temporal power spectrum of $e^{-2\pi i\mathbf{k}\cdot\xi(t)}$.

We have previously showed that the same power spectrum $P_L(k, \omega)$ can be written as

$$P_L(k, \omega) = P_L(k) Q(k, \omega), \tag{13}$$

where $Q(k, \omega)$ is the Fourier transform in both space and time of $q(x, \tau)$,

$$q(x, \tau) = \int p(\xi(t_0 + \tau) = x_1 + x, \xi(t_0) = x_1) dx_1. \tag{14}$$

From both Eq. (10) and Eq. (13), it must follow that $P_E(k, \omega) = Q(k, \omega)$. This can be shown as follows. $P_E(k, \omega)$ can be obtained by Fourier transform an autocorrelation function $C_E(k, \tau)$ in time, where

$$C_E(k, \tau) = \left\langle e^{-2\pi i k \xi(t_0)} e^{2\pi i k \xi(t_0+\tau)} \right\rangle_{t_0}. \tag{15}$$

Assuming that the eye movement is characterized by a probability function $p(\xi(t_0) = x_1, \xi(t_0+\tau) = x_2)$, we have

$$C_E(k, \tau) = \left\langle \int dx_1 \int dx_2 e^{-2\pi i k x_1} e^{2\pi i k x_2} p(\xi(t_0) = x_1, \xi(t_0 + \tau) = x_2) \right\rangle_{t_0}. \tag{16}$$

If we perform an inverse Fourier transform on $C_E(k, \tau)$ in space, we obtain

$$\begin{aligned} C_E(x, \tau) &= \left\langle \int dx_1 \int dx_2 \int dk e^{-2\pi i k(x_1 - x_2 - x)} p(\xi(t_0) = x_1, \xi(t_0 + \tau) = x_2) \right\rangle_{t_0} \\ &= \left\langle \int dx_1 \int dx_2 \delta(x_1 - x_2 - x) p(\xi(t_0) = x_1, \xi(t_0 + \tau) = x_2) \right\rangle_{t_0} \\ &= \left\langle \int dx_1 p(\xi(t_0) = x_1, \xi(t_0 + \tau) = x_1 - x) \right\rangle_{t_0} \\ &= q(x, \tau), \end{aligned} \tag{17}$$

assuming t_0 can be neglected in the long-time limit.

Since $P_E(k, \omega)$ is the Fourier transform of $C_E(x, \tau)$, and $Q(k, \omega)$ is the Fourier transform of $q(x, \tau)$, with $C_E(x, \tau) = q(x, \tau)$, we know that $P_E(k, \omega) = Q(k, \omega)$.