Effects of eye movements on spectra of movies

(with M. Rucci)

In v05, we add a section that focuses on single saccades NOT as point processes. Also a comment about drifting gratings at end of "basic calculation" section.

In v06, we add a section about fractional Brownian motion.

Setup

The goal is to calculate the effects of eye movements on the autocorrelation of a spatiotemporal stimulus. That is, given a spatiotemporal stimulus $S(x,t)$ (with mean 0) with known autocorrelation C_S, we want to calculate C_M, the autocorrelation of the spatiotemporal pattern on the retina, after considering eye movements.

Say the eye movement process has displacements $\xi(t)$, characterized by a known $p(\xi(t_1)=x_1\,|\,\xi(t_0)=x_0)$.

We assume that the eye movements described by $\xi(t)$ are independent of the image. But to allow for the possibility that saccades occur at "special" times in the image sequence, we assume that a saccade has occurred at time 0, and allow for the possibility that the saccades may depend on image statistics. We formalize this dependence by characterizing the stimulus by its autocorrelation at a time t_0 after the saccade:

$$C_S(x,\tau;t_0)=\left\langle S(x_0,t_0)S(x_0+x,t_0+\tau)\right\rangle_{x_0}. \tag{1.1.1}$$

The autocorrelation on the retina is then

$$C_M(x,\tau;t_0)=\left\langle S(x_0+\xi(t_0),t_0)S(x_0+x+\xi(t_0+\tau),t_0+\tau)\right\rangle_{x_0}. \tag{1.1.2}$$

Averaged over all times, the autocorrelation is

$$C_M(x,\tau)=\left\langle S(x_0+\xi(t_0),t_0)S(x_0+x+\xi(t_0+t),t_0+\tau)\right\rangle_{x_0,t_0}. \tag{1.1.3}$$

We also expect that this will be the behavior of eq. (1.1.2) for large t_0. For large t_0, or, averaged over all t_0 (which is dominated by t_0 large), we anticipate that eq. (1.1.3) will depend only on the average autocorrelation of $S(x,t)$,

$$C_S(x,\tau)=\left\langle S(x_0,t_0)S(x_0+x,t_0+\tau)\right\rangle_{x_0,t_0}, \tag{1.1.4}$$

and that the time-dependent quantity (eq. (1.1.1)) won't be necessary.

Basic calculation

We calculate eq. (1.1.2), and then consider its behavior for large t_0. Eq. (1.1.2) can be written:

$$C_M(x,t;t_0)=\iint\left\langle S(x_0+x_1,t_0)S(x_0+x+x_2,t_0+\tau)\,p(\xi(t_0)=x_1)\,p(\xi(t_0+\tau)=x_2\,|\,\xi(t_0)=x_1)\right\rangle_{x_0}dx_1dx_2, \tag{2.1.1}$$

which states that the autocorrelation on the retina is a sum of contributions over all eye movement paths, in which the eye starts at position x_1 at time t_0, and ends up at position x_2 at time t_0+t. We re-express this in terms of an initial position on the image, $y=x_0+x_1$, and the displacement due to the eye movement, $\Delta x=x_2-x_1$:

$$C_M(x,t;t_0)=\iint \left\langle S(y,t_0)S(y+x+\Delta x,t_0+\tau)\right) p(\xi(t_0)=x_1)\,p(\xi(t_0+\tau)=x_1+\Delta x\,|\,\xi(t_0)=x_1)\rangle_y\, dx_1 d\Delta x \quad .(2.1.2)$$

Note that the S-terms and the p-terms are independent. The S-terms do not depend on x_1, which is the position of the eyes at time t_0 (because the movie is considered to be statistically spatially homogeneous). The p-terms, which describe the eye trajectory, do not depend on y, which is the initial position on the image – and the eye movements described by $\xi(t)$ are assumed not to depend on the image. (Maybe this is not true at the fovea.)

What this means is that we can integrate out initial eye position, x_1, and replace that factor by a quantity that describes the eye movement distribution:

$$q(\Delta x,\tau;t_0)=\int p(\xi(t_0)=x_1)\,p(\xi(t_0+\tau)=x_1+\Delta x\,|\,\xi(t_0)=x_1)dx_1,$$

which is the probability that the eyes move by Δx between t_0 and $t=t_0+\tau$ (summed over all starting locations at t_0). Equivalently, making use of the relationship between a conditional and a joint probability,

$$q(\Delta x,\tau;t_0)=\int p(\xi(t_0+\tau)=x_1+\Delta x,\xi(t_0)=x_1)dx_1. \quad (2.1.3)$$

Thus, eq. (2.1.2) becomes

$$C_M(x,\tau;t_0)=\int \left\langle S(y,t_0)S(y+x+\Delta x,t_0+\tau)q(\Delta x,\tau;t_0)\right\rangle_y d\Delta x. \quad (2.1.4)$$

Put another way, C_M and C_S are related by convolution in space:

$$C_M(\bullet,\tau;t_0)=C_S(\bullet,\tau;t_0)*q(\bullet,\tau;t_0). \quad (2.1.5)$$

We use a single-tilde to indicate Fourier transform in the space domain, and a double-tilde to indicate Fourier transformation in space and time, or just in time.

With $\tilde{C}_M(k,\tau;t_0)=\int_{-\infty}^{\infty} C_M(x,\tau;t_0)e^{-ikx}dx$, $\tilde{C}_S(k,\tau;t_0)=\int_{-\infty}^{\infty} C_S(x,\tau;t_0)e^{-ikx}dx$, and

$\tilde{q}(k,\tau;t_0)=\int_{-\infty}^{\infty} q(x,\tau;t_0)e^{-ikx}dx$, it follows that

$$\tilde{C}_M(k,\tau;t_0)=\tilde{C}_S(k,\tau;t_0)\tilde{q}(k,\tau;t_0). \quad (2.1.6)$$

In the long-time limit, the dependence on t_0 can be neglected:

$$\tilde{C}_M(k,\tau)=\tilde{C}_S(k,\tau)\tilde{q}(k,\tau). \quad (2.1.7)$$

The multiplication point-by-point in τ is equivalent to convolving in temporal frequency, ω.

$$\tilde{\tilde{C}}_M(k,\omega)=\tilde{\tilde{C}}_S(k,\omega)*_\omega \tilde{\tilde{q}}(k,\omega) \quad (2.1.8)$$

where

$$\tilde{\tilde{q}}(k,\omega)=\int_{-\infty}^{\infty}\tilde{q}(k,\tau)\exp(-i\omega\tau)d\tau \tag{2.1.9}$$

and $*_\omega$ indicates convolution in the temporal frequency domain.

There's a faster way to arrive directly at eq. (2.1.8), if we simply assume that the movie and the eye movements are independent. We can calculate the (k,τ) representation for an instance of movie and eye movements – it is pointwise multiplication (since it is convolution in space, and multiplication in time), and then calculate the covariance over the set of movies and the set of eye movements, and, since these are independent, it factors.

Comment re measuring spatiotemporal receptive fields with gratings

A propos discussions with M. Rucci and B. Shapley, May 17, 2016. Note that if the stimulus consists of only a single spatiotemporal frequency, $\tilde{\tilde{C}}_S(k,\omega)=\delta(k-k_0)\delta(\omega-\omega_0)$, then (2.1.8) becomes

$$\tilde{\tilde{C}}_M(k,\omega)=\tilde{\tilde{C}}_S(k,\omega)*_\omega\tilde{\tilde{q}}(k,\omega)=\delta(k-k_0)\tilde{\tilde{q}}(k_0,\omega+\omega_0). \tag{2.1.10}$$

This has implications for measuring a neuron's spatial transfer function with drifting gratings. Say the neuron is separable, i.e., that its transfer function is some $L(k,\omega)=K(k)W(\omega)$. The power in the response to a drifting grating is given by

$$R(k_0,\omega_0)=\iint\tilde{\tilde{C}}_M(k,\omega)\left|L(k,\omega)\right|^2dkd\omega$$

$$=\iint\tilde{\tilde{q}}(k_0,\omega+\omega_0)\delta(k-k_0)\left|K(k)W(\omega)\right|^2dkd\omega \tag{2.1.11}$$

$$=\left|K(k_0)\right|^2\int\tilde{\tilde{q}}(k_0,\omega+\omega_0)\left|W(\omega)\right|^2d\omega$$

So even though the neuron itself is spatiotemporally separable, the measured response power, $R(k_0,\omega_0)$, is influenced by the coupling based on $\tilde{\tilde{q}}(k,\omega)$.

Some special cases

We work out these equations for some simple models of image statistics and eye movements. In both cases, we use common technology from the theory of renewal processes. Say $p(t)$ is the renewal density, i.e., $p(t)\Delta t$ is the probability that the first event that follows an event at time T will occur between T and $T+t+\Delta t$. (In a renewal process, by definition, the event probability depends only on the time since the last event, and is independent of absolute time T.) Our goal is to calculate $K_N(t)$, the probability that there are N events in an interval of length t, independent of whether or not there is an event at time 0.

Distribution of the number of expected events for a renewal process

We will do this in the Fourier domain, so we make heavy use of

$$\tilde{p}(\omega)=\int_0^{\infty}e^{-i\omega t}p(t)dt. \tag{3.1.1}$$

Note that $\tilde{p}(0)=1$ because of normalization, and that, since

$$\tilde{\tilde{p}}'(\omega) = -\int_0^\infty ite^{-i\omega t} p(t)dt\,, \tag{3.1.2}$$

that

$$i\tilde{\tilde{p}}'(0) = \int_0^\infty tp(t)dt\,, \tag{3.1.3}$$

the mean interval.

We will need $p_{within}(t)$, the probability that a randomly-chosen time is within an interval of length t. This is proportional to $tp(t)$. Because it must be normalized,

$$p_{within}(t) = \frac{tp(t)}{\int_0^\infty \tau p(\tau)d\tau}\,. \tag{3.1.4}$$

For $N > 0$, $K_N(t)$ (the probability that an interval of length *t* contains exactly *N* events) has a contribution for each event sequence at times $0 < t_1 < \ldots < t_N < t$. For the intermediate intervals ((t_{n-1}, t_n) $1 < n \le N$), the probability of an event at time t_n given an event at time t_{n-1} is $p(t_n - t_{n-1})$. For the first and last intervals, we need $p_{first}(t)$, the probability that the first event after an arbitrary time occurs at a time t, and $p_{last}(t)$, the probability that no event occurs within time *t* following an event. And for $N = 0$, we need $p_{none}(t)$, the probability that there is no event in a randomly chosen interval of length *t*.

With these quantities, we have (for $N > 0$)

$$K_N(t) = \int_{t_1 < \cdots < t_n} \ldots \int p_{first}(t_1)p(t_2 - t_1)\ldots p(t_n - t_{n-1})p_{last}(t - t_n)dt_1 \ldots dt_n\,. \tag{3.1.5}$$

This is a convolution, so

$$\tilde{\tilde{K}}_N(\omega) = \tilde{\tilde{p}}_{first}(\omega)\left(\tilde{\tilde{p}}(\omega)\right)^{N-1} \tilde{\tilde{p}}_{last}(\omega)\,. \tag{3.1.6}$$

For the special case of $N = 0$,

$$\tilde{\tilde{K}}_0(\omega) = \tilde{\tilde{p}}_{none}(\omega)\,. \tag{3.1.7}$$

To calculate these quantities, it is easiest to begin with $p_{last}(t)$. It is the total probability that the next event is at least at a time *t* in the future. Therefore,

$$p_{last}(t) = \int_t^\infty p(\tau)d\tau\,. \tag{3.1.8}$$

Thus (noting that $p_{last}(0) = 1$ but $p_{last}(t) = 0$ for $t < 0$, so there's a unit jump at $t = 0$),

$$p'_{last}(t) = \delta(t) - p(t)\,. \tag{3.1.9}$$

Since

$$p(t) = \frac{1}{2\pi}\int_0^\infty \tilde{\tilde{p}}(\omega)e^{i\omega t}d\omega\,, \tag{3.1.10}$$

eq. (3.1.9) is equivalent to

$$i\omega\tilde{\tilde{p}}_{last}(\omega) = \tilde{\tilde{\delta}}(\omega) - \tilde{\tilde{p}}(\omega)\,, \tag{3.1.11}$$

And since $\tilde{\tilde{\delta}}(\omega)=1$, it follows from (3.1.11) that

$$\tilde{\tilde{p}}_{last}(\omega)=\frac{1}{i\omega}(1-\tilde{\tilde{p}}(\omega)). \tag{3.1.12}$$

From l'Hôpital's rule,

$$\lim_{\omega\to 0}\tilde{\tilde{p}}_{first}(\omega)=i\tilde{\tilde{p}}'(0), \tag{3.1.13}$$

the mean interval, and, the Taylor expansion near 0 is

$$\tilde{\tilde{p}}_{first}(\omega)=i\tilde{\tilde{p}}'(0)+i\frac{\omega}{2}\tilde{\tilde{p}}''(0)+\dots. \tag{3.1.14}$$

To calculate $p_{first}(t)$, we note that it is contains a contribution from all intervals of length $\tau\geq t$. It is the probability that time 0 is inside such an interval times the probability that, conditional on being in this interval, that it is $t-\tau$ from the end. The former is ($p_{within}(t)$, eq. (3.1.4))); the latter is $1/\tau$. Therefore,

$$p_{first}(t)=\int_t^\infty \frac{1}{\tau}p_{within}(\tau)d\tau=\frac{\int_t^\infty p(\tau)d\tau}{\int_0^\infty \tau p(\tau)d\tau}. \tag{3.1.15}$$

The numerator is $p_{last}(t)$ and the denominator is the mean interval, (3.1.3). Therefore,

$$\tilde{\tilde{p}}_{first}(\omega)=\frac{\tilde{\tilde{p}}_{last}(\omega)}{i\tilde{\tilde{p}}'(0)}=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega\left(i\tilde{\tilde{p}}'(0)\right)}=\frac{1-\tilde{\tilde{p}}(\omega)}{-\omega\tilde{\tilde{p}}'(0)}. \tag{3.1.16}$$

We could also have seen the relationship of $p_{first}(t)$ to $p_{last}(t)$ (i.e., $p_{last}(t)/p_{first}(t)$ is the mean interval) by recognizing that the time series is invertible, and that the only difference between $p_{first}(t)$ and $p_{last}(t)$ is that $p_{last}(t)$ is conditioned on the occurrence of an event, which has probability per unit time of (1/mean interval). Note that

$$\lim_{\omega\to 0}\tilde{\tilde{p}}_{first}(\omega)=1. \tag{3.1.17}$$

A similar argument yields $p_{none}(t)$. Given that a randomly-chosen time is within an interval of length τ, the probability that the following *t* seconds stays within the interval (and therefore, contains no events) is $(t-\tau)/\tau$. Therefore,

$$p_{none}(t)=\int_t^\infty \frac{\tau-t}{\tau}p_{within}(\tau)d\tau=\frac{\int_t^\infty (\tau-t)p(\tau)d\tau}{\int_0^\infty \tau p(\tau)d\tau}. \tag{3.1.18}$$

From this it follows that

$$p'_{none}(t)=\delta(t)-p_{first}(t) \tag{3.1.19}$$

and that

$$\tilde{\tilde{p}}_{none}(\omega)=\frac{1}{i\omega}(1-\tilde{\tilde{p}}_{first}(\omega))=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega^2\tilde{\tilde{p}}'(0)}+\frac{1}{i\omega}. \tag{3.1.20}$$

From the first part of eq. (3.1.14) and L'hopital's rule, it follows that

$$\lim_{\omega\to 0}\tilde{\tilde{p}}_{none}(\omega) = i\tilde{\tilde{p}}'_{first}(0) = i\frac{i}{2}p''(0) = -\frac{1}{2}p''(0)\,. \tag{3.1.21}$$

Eq. (3.1.20) can be checked from (3.1.6) and (3.1.7), since we should have $\sum_{N=0}^{\infty}\tilde{\tilde{K}}_N(\omega) = \frac{1}{i\omega}$ (the Fourier transform of the Heaviside function):

$$\begin{aligned}\sum_{N=0}^{\infty}\tilde{\tilde{K}}_N(\omega) &= \tilde{\tilde{p}}_{none}(\omega) + \sum_{N=1}^{\infty}\tilde{\tilde{p}}_{first}(\omega)\left(\tilde{\tilde{p}}(\omega)\right)^{N-1}\tilde{\tilde{p}}_{last}(\omega) = \tilde{\tilde{p}}_{none}(\omega) + \frac{\tilde{\tilde{p}}_{first}(\omega)\tilde{\tilde{p}}_{last}(\omega)}{1-\tilde{\tilde{p}}(\omega)} \\ &= \frac{1}{i\omega} - \frac{1}{i\omega}\tilde{\tilde{p}}_{first}(\omega) + \frac{\tilde{\tilde{p}}_{first}(\omega)\tilde{\tilde{p}}_{last}(\omega)}{1-\tilde{\tilde{p}}(\omega)} = \frac{1}{i\omega}\end{aligned}, \tag{3.1.22}$$

with the last equality following from (3.1.12).

Poisson process

For a Poisson process with rate λ, the renewal density is

$$p^{Poisson}(t) = \lambda e^{-\lambda t}\,, \tag{3.2.1}$$

its Fourier transform is

$$\tilde{\tilde{p}}^{Poisson}(\omega) = \frac{1}{1+i\omega/\lambda}\,, \tag{3.2.2}$$

and

$$\tilde{\tilde{p}}'^{Poisson}(0) = -i/\lambda\,. \tag{3.2.3}$$

From eq. (3.1.12),

$$\tilde{\tilde{p}}_{last}^{Poisson}(\omega) = \frac{1}{\lambda(1+i\omega/\lambda)} = \frac{1}{\lambda+i\omega} = \frac{1}{\lambda}\tilde{\tilde{p}}^{Poisson}(\omega)\,. \tag{3.2.4}$$

From eq. (3.1.16),

$$\tilde{\tilde{p}}_{first}^{Poisson}(\omega) = \frac{1}{1+i\omega/\lambda} = \tilde{\tilde{p}}^{Poisson}(\omega)\,. \tag{3.2.5}$$

From eq. (3.1.20),

$$\tilde{\tilde{p}}_{none}^{Poisson}(\omega) = \frac{1}{\lambda(1+i\omega/\lambda)} = \frac{1}{\lambda}\tilde{\tilde{p}}^{Poisson}(\omega)\,. \tag{3.2.6}$$

Stimulus autocorrelation

We consider a few model cases, and assume that the images are statistically homogeneous. We use $c_S(x)$ to denote the spatial autocorrelation of the image ensemble, i.e.,

$$c_S(x) = \left\langle S(x_0)S(x_0+x)\right\rangle_{x_0}\,, \tag{4.1.1}$$

and

$$\tilde{c}_S(k) = \int_{-\infty}^{\infty} c_S(x)e^{-ikx}dx\,. \tag{4.1.2}$$

A single still image

When the stimulus is a single still image, $S(x,t) = S(x)$, then the spatiotemporal autocorrelation only depends on the spatial displacement. So

$$C_S(x,\tau) = c_S(\tau) \quad (4.2.1)$$

and its Fourier transform is given by

$$\tilde{\tilde{C}}_S(k,\omega) = 2\pi \tilde{c}_S(k)\delta(\omega)\,. \quad (4.2.2)$$

For this, the convolution in eq. (2.1.8) becomes trivial:

$$\tilde{\tilde{C}}_M(k,\omega) = \tilde{c}_S(k)\tilde{\tilde{q}}(k,\omega)\,. \quad (4.2.3)$$

Randomly changing snapshots

Assume that the stimulus consists of a static image that changes according to a renewal process with renewal density $p(t)$, and each image is uncorrelated. If there is no change in the image within the time lag of the autocorrelation, the result of the above section applies. This happens with probability $K_0(\tau)$. Otherwise, since the images are uncorrelated, the autocorrelation is zero. Thus (taking both positive and negative times into account)

$$C_S(x,\tau;t_0) = c_S(x)K_0(|\tau|)\,. \quad (4.3.1)$$

The Fourier transform of $K_N(|\tau|)$ contains a contribution $\tilde{\tilde{K}}_N(\omega)$ for $\tau > 0$ and a contribution $\tilde{\tilde{K}}_N(-\omega)$ for $\tau < 0$,

$$\tilde{\tilde{C}}_S(k,\omega) = \tilde{c}_S(k)\left(\tilde{\tilde{p}}_{none}(\omega) + \tilde{\tilde{p}}_{none}(-\omega)\right). \quad (4.3.2)$$

We can view this either as changes in the image, or as saccades at times determined by $p(t)$. For the latter, one can use a gamma distribution

$$p^{gamma}(t) = \frac{(a\lambda)^a}{\Gamma(a)} t^{a-1} e^{-a\lambda t}\,. \quad (4.3.3)$$

This has a mean rate λ, a mean interval $1/\lambda$, and a variance for the mean interval of $1/a\lambda^2$. Since it is the *a*-fold convolution of a Poisson process of rate $a\lambda$, its Fourier transform is

$$\tilde{\tilde{p}}^{gamma}(\omega) = \left(\frac{1}{1 + i\omega/(a\lambda)}\right)^a\,. \quad (4.3.4)$$

The eye movement process

One dimension, Brownian

For a Brownian process, $q(\Delta x, \tau; t_0)$ is independent of t_0. We assume that the eye position corresponds to a diffusion governed by

$$\frac{\partial}{\partial t} q(x,t) = D_{Brownian} \frac{\partial^2}{\partial x^2} q(x,t)\,. \quad (5.1.1)$$

Via standard techniques, this leads to

$$\tilde{q}(k,\tau)_{Brownian} = \exp(-k^2 D_{Brownian}|\tau|)\,. \quad (5.1.2)$$

That is, the eye position, after time τ, has a probability distribution whose variance is $2D_{Brownian}|\tau|$, namely,

$$q(x,\tau)=\frac{1}{\sqrt{4\pi D_{Brownian}\tau}}\exp(-\frac{x^2}{4D_{Brownian}\tau}). \tag{5.1.3}$$

Since $\tilde{q}(k,\tau)=\int_{-\infty}^{\infty} q(x,\tau)e^{-ikx}dx$,

$$\left\langle x^2\right\rangle=\int_{-\infty}^{\infty} x^2 q(x,\tau)dx=-\frac{\partial^2}{\partial k^2}\tilde{q}(k,\tau)\Big|_{k=0}. \tag{5.1.4}$$

Thus, the variance after time τ can be directly found from eq. (5.1.2):

$$\left\langle x^2\right\rangle=2D_{Brownian}\tau. \tag{5.1.5}$$

To see the effects on static images, we apply eq. (2.1.8). We need the spatiotemporal transform of eq. (5.1.2):

$$\tilde{\tilde{q}}_{Brownian-1D}(k,\omega)=\int_{-\infty}^{\infty}\exp(-k^2D_{Brownian}\left|\tau\right|)\exp(-i\omega\tau)d\tau c \tag{5.1.6}$$

A standard result is

$$\int_{-\infty}^{\infty}\exp(-b\left|\tau\right|)\exp(-i\omega\tau)d\tau=\frac{2b}{b^2+\omega^2}. \tag{5.1.7}$$

So with $b=k^2D_{Brownian}$,

$$\tilde{\tilde{q}}_{Brownian-1D}(k,\omega)=2\frac{k^2D_{Brownian}}{k^4D_{Brownian}^2+\omega^2} \tag{5.1.8}$$

For $b<<\omega$, i.e., $\left|k\right|<<\sqrt{\omega/D_{Brownian}}$,

$$\tilde{\tilde{q}}_{Brownian-1D}(k,\omega)\approx 2k^2D_{Brownian}\omega^{-2}, \tag{5.1.9}$$

which (in eq.(4.2.3)) neutralizes a power spectrum of k^{-2}.

The optimal D

With the idea of asking whether fixational eye movements can be tuned to task, let's find the value of D where $\tilde{\tilde{q}}(k,\omega)=2\frac{k^2D_{Brownian}}{k^4D_{Brownian}^2+\omega^2}$ is maximal.

$$\begin{aligned}\frac{\partial}{\partial D}\log\tilde{\tilde{q}}(k,\omega)&=\frac{\partial}{\partial D}\log\left(2\frac{k^2D}{k^4D^2+\omega^2}\right)\\&=\frac{\partial}{\partial D}\log\left(D\right)-\frac{\partial}{\partial D}\left(k^4D^2+\omega^2\right)\\&=\frac{1}{D}-\frac{2k^4D}{k^4D^2+\omega^2}\end{aligned}. \tag{5.1.10}$$

So $\frac{\partial}{\partial D}\tilde{\tilde{q}}(k,\omega)=0$ occurs when $2k^4D^2=k^4D^2+\omega^2$, i.e., when $k^4D^2=\omega^2$, i.e., when $D=\omega/k^2$. This means we predict that partial stabilization will improve sensitivity if $D>\omega/k^2$.

One dimension, Brownian, with drift

Let's say that the mean eye position moves at a fixed velocity c. So now

$$q_{BrownianDrift}(\Delta x,\tau;t_0)=q_{Brownian}(\Delta x-c\tau,\tau;t_0) \quad (5.1.11)$$

where $q_{Brownian}$ is given by eq. (5.1.1). Then

$$\tilde{q}_{BrownianDrift}(k,\tau)=\int_{-\infty}^{\infty} q_{BrownianDrift}(x,\tau)e^{-ikx}dx$$

$$=\int_{-\infty}^{\infty} q_{Brownian}(x-c\tau,\tau)e^{-ikx}dx=\int_{-\infty}^{\infty} q_{Brownian}(u,\tau)e^{-ik(u+c\tau)}du\,. \quad (5.1.12)$$

$$=e^{-ikc\tau}\tilde{q}_{Brownian}(k,\tau)=e^{-ikc\tau}\exp(-k^2D_{Brownian}\left|\tau\right|)$$

Calculating as above,

$$\tilde{\tilde{q}}_{BrownianDrift-1D}(k,\omega)=\int_{-\infty}^{\infty}\exp(-ikc\tau-k^2D_{Brownian}\left|\tau\right|)\exp(-i\omega\tau)d\tau$$

$$=\int_{-\infty}^{\infty}\exp(-k^2D_{Brownian}\left|\tau\right|)\exp(-i(\omega+kc)\tau)d\tau=\tilde{\tilde{q}}_{Brownian-1D}(k,\omega+kc)\,. \quad (5.1.13)$$

$$=2\frac{k^2D_{Brownian}}{k^4D_{Brownian}^2+(\omega+kc)^2}$$

The no-drift asymptotic analysis (for $\left|k\right|<<\sqrt{\omega/D_{Brownian}}$, that the second term in the denominator dominates) still holds provided that also

$$\left|\omega+kc\right|>>k^2D_{Brownian}\,.$$

(5.1.14).

Not surprisingly, in the neighborhood of $\omega+kc=0$, i.e., when $k=-\omega/c$, which corresponds to spatiotemporal components that move synchronously with the drift, there is a "resonance:"

$$\tilde{\tilde{q}}_{BrownianDrift-1D}(k,-kc)\simeq\frac{2}{k^2D_{Brownian}}\,.$$

Two dimensions, Brownian

To match the 1-D case, we set this up so that the variance at time τ is $2D\tau$ (expected distance squared in the plane). Since this is a sum of an *x*-variance and a *y*-variance, we'd want the *x*-variance and *y*-variance each to be $D\tau$. So, the relationship to the one-dimensional problem is $D_{Brownian}=D/2$, and the diffusion law is

$$\frac{\partial}{\partial t}q(x,y,t)=D_{Brownian}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)q(x,y,t)=\frac{1}{2}D\nabla^2q(x,y,t)\,. \quad (5.1.15)$$

Diffusion along each coordinate is independent. So, eq. (5.1.2) yields

$$\tilde{q}(k_x,k_y,\tau)=e^{-(k_x{}^2+k_y{}^2)D\left|\tau\right|/2} \quad (5.1.16)$$

and eq. (5.1.3) yields

$$q(x,y,\tau)=\left(\frac{1}{\sqrt{2\pi D\tau}}\exp(-\frac{x^2}{2D\tau})\right)\left(\frac{1}{\sqrt{2\pi D\tau}}\exp(-\frac{y^2}{2D\tau})\right)=\frac{1}{2\pi D\tau}\exp(-\frac{x^2+y^2}{2D\tau}), \qquad (5.1.17)$$

and we can check that

$$\left\langle x^2+y^2\right\rangle=-\left(\frac{\partial^2}{\partial k_x^{\ 2}}+\frac{\partial^2}{\partial k_x^{\ 2}}\right)\tilde{q}(k_x,k_y,\tau)\Big|_{k=0}=2D\left|\tau\right|. \qquad (5.1.18)$$

Again using eq. (5.1.7) with $b=\left(\frac{k_x^{\ 2}+k_y^{\ 2}}{2}\right)D$, the spatiotemporal transform is

$$\tilde{\tilde{q}}_{Brownian-2D}(k_x,k_y,\omega)=\int_{-\infty}^{\infty}\tilde{q}(k_x,k_y,\tau)e^{-i\omega\tau}d\omega=\frac{(k_x^{\ 2}+k_y^{\ 2})D}{(k_x^{\ 2}+k_y^{\ 2})^2D^2/4+\omega^2}. \qquad (5.1.19)$$

For small k or short times, i.e., for $k_x^2+k_y^2<<2\omega/D$ (equivalently, $\left|k\right|<<\sqrt{\omega/D_{Brownian}}$) this is approximately

$$\tilde{\tilde{q}}_{Brownian-2D}(k_x,k_y,\omega)\approx(k_x^{\ 2}+k_y^{\ 2})D\omega^{-2}, \qquad (5.1.20)$$

which (in eq.(4.2.3)) also neutralizes a power spectrum of k^{-2}.

Note that had we considered a separate D_x and D_y, we'd have found

$$\tilde{\tilde{q}}_{Brownian-2D}(k_x,k_y,\omega)\approx(k_x^{\ 2}D_x+k_y^{\ 2}D_y)\omega^{-2}, \qquad (5.1.21)$$

so, not surprisingly, if diffusion only occurs in one dimension ($D_y=0$), the neutralization only applies to Fourier components that vary in the other dimension.

Fractional Brownian motion

Generalize (5.1.16) to

$$\tilde{q}(\left|\vec{k}\right|,\tau)=e^{-\left|\vec{k}\right|^2D\left|\tau\right|^h/2}. \qquad (5.1.22)$$

We'd like to compute, or at least estimate,

$$\tilde{\tilde{q}}_{fbr}(\left|\vec{k}\right|,\omega)=\int_{-\infty}^{\infty}\tilde{q}(\left|\vec{k}\right|)e^{-i\omega\tau}dt. \qquad (5.1.23)$$

Put $b=\left|\vec{k}\right|^2D$. Then $\tilde{q}(\left|\vec{k}\right|,\tau)=e^{-b\left|\tau\right|^h/2}$ and $\tilde{\tilde{q}}_{fbr}(\left|\vec{k}\right|,\omega)=Z(b,\omega)$, where

$$Z(b,\omega)=\int_{-\infty}^{\infty}e^{-b\left|\tau\right|^h/2}e^{-i\omega\tau}dt. \qquad (5.1.24)$$

Note a kind of scaling:

$$Z(b,\omega)=\int_{-\infty}^{\infty}e^{-b\left|\tau\right|^h/2}e^{-i\omega\tau}d\tau=b^{-1/h}\int_{-\infty}^{\infty}e^{-\left|t\right|^h/2}e^{-i\omega b^{-1/h}t}dt=b^{-1/h}Z(1,\omega b^{-1/h}). \qquad (5.1.25)$$

We ***hypothesize*** (based on eyemov_spec_fbm.m) that for large ω ($\omega >> b$),

$$Z(b,\omega) \sim A(b,h)\omega^{-h-1}. \tag{5.1.26}$$

This makes sense from the kind of "fractional derivative" discontinuity that (5.1.22) has near 0. Assuming this is the case, then looking at (5.1.25) for large ω:

$$Z(b,\omega) = b^{-1/h} Z(1,\omega b^{-1/h}) \sim b^{-1/h} A(1,h)(\omega b^{-1/h})^{-h-1} = A(1,h)\omega^{-h-1} b = A(1,h)\omega^{-h-1} \left|\vec{k}\right|^2 D. \tag{5.1.27}$$

That is, for sufficiently large temporal frequencies ω

$$\tilde{\tilde{q}}_{fbr}(\left|\vec{k}\right|,\omega) \sim A(1,h)\omega^{-h-1} \left|\vec{k}\right|^2 D, \tag{5.1.28}$$

which yields whitening, since, integrating over the same (high) range of ω's yields results that are proportional to $\left|\vec{k}\right|^2$.

Bazant ("Lecture 22", https://ocw.mit.edu/courses/mathematics/18-366-random-walks-and-diffusion-fall-2006/lecture-notes/lec22_neville.pdf) in eqs. 23 to 29 shows that

$$L_\alpha(a,x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|\tau|^\alpha} e^{ix\tau} d\tau \sim \frac{a\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi |x|^{1+\alpha}} \tag{5.1.29}$$

so

$$Z(b,\omega) = 2\pi L_h(\frac{b}{2},\omega) \sim \frac{bh\Gamma(h)\sin\frac{\pi h}{2}}{|\omega|^{1+h}}. \tag{5.1.30}$$

So the hypothesis (5.1.26) holds with

$$A(b,h) = b\Gamma(h+1)\sin\frac{\pi h}{2}. \tag{5.1.31}$$

The validity of the asymptotic is illustrated in eyemov_spec_fbm.m.

Abrupt refixations within random steps

We can use the renewal process analysis to analyze another case: saccades with renewal density $p(t)$, and in which the step is drawn from a random distribution $r(x,y)$ with Fourier transform $\tilde{r}(k_x,k_y)$. If exactly *N* steps are taken (which has a probability $K_N(\tau)$), then the distribution of displacements has Fourier transform $\left(\tilde{r}(k_x,k_y)\right)^N$. Therefore (with "c.c." for complex conjugate, i.e., substituting $-\omega$ for ω),

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega) = \sum_0^N \tilde{\tilde{K}}_N(\omega)\left(\tilde{r}(k_x,k_y)\right)^N + c.c. \tag{5.2.1}$$

or,

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega)=\tilde{\tilde{p}}_{none}(\omega)+\sum_{1}^{N}\tilde{\tilde{p}}_{first}(\omega)\left(\tilde{\tilde{p}}(\omega)\right)^{N-1}\tilde{\tilde{p}}_{last}(\omega)\left(\tilde{r}(k_x,k_y)\right)^{N}+c.c. \tag{5.2.2}$$

or,

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega)=\tilde{\tilde{p}}_{none}(\omega)+\frac{\tilde{\tilde{p}}_{first}(\omega)\tilde{\tilde{p}}_{last}(\omega)\tilde{r}(k_x,k_y)}{1-\tilde{\tilde{p}}(\omega)\tilde{r}(k_x,k_y)}+c.c. \tag{5.2.3}$$

or,

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega)=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega^2\tilde{\tilde{p}}'(0)}-\frac{\left(1-\tilde{\tilde{p}}(\omega)\right)^2\tilde{r}(k_x,k_y)}{i\omega^2\tilde{\tilde{p}}'(0)\left(1-\tilde{\tilde{p}}(\omega)\tilde{r}(k_x,k_y)\right)}+c.c. \tag{5.2.4}$$

or,

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega)=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega^2\tilde{\tilde{p}}'(0)}-\frac{\left(1-\tilde{\tilde{p}}(\omega)\right)^2\tilde{r}(k_x,k_y)}{i\omega^2\tilde{\tilde{p}}'(0)\left(1-\tilde{\tilde{p}}(\omega)\tilde{r}(k_x,k_y)\right)}+c.c. \tag{5.2.5}$$

or,

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega)=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega^2\tilde{\tilde{p}}'(0)}\left(1-\frac{\left(1-\tilde{\tilde{p}}(\omega)\right)\tilde{r}(k_x,k_y)}{\left(1-\tilde{\tilde{p}}(\omega)\tilde{r}(k_x,k_y)\right)}\right)+c.c. \tag{5.2.6}$$

or,

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega)=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega^2\tilde{\tilde{p}}'(0)}\frac{1-\tilde{r}(k_x,k_y)}{\left(1-\tilde{\tilde{p}}(\omega)\tilde{r}(k_x,k_y)\right)}+c.c. \tag{5.2.7}$$

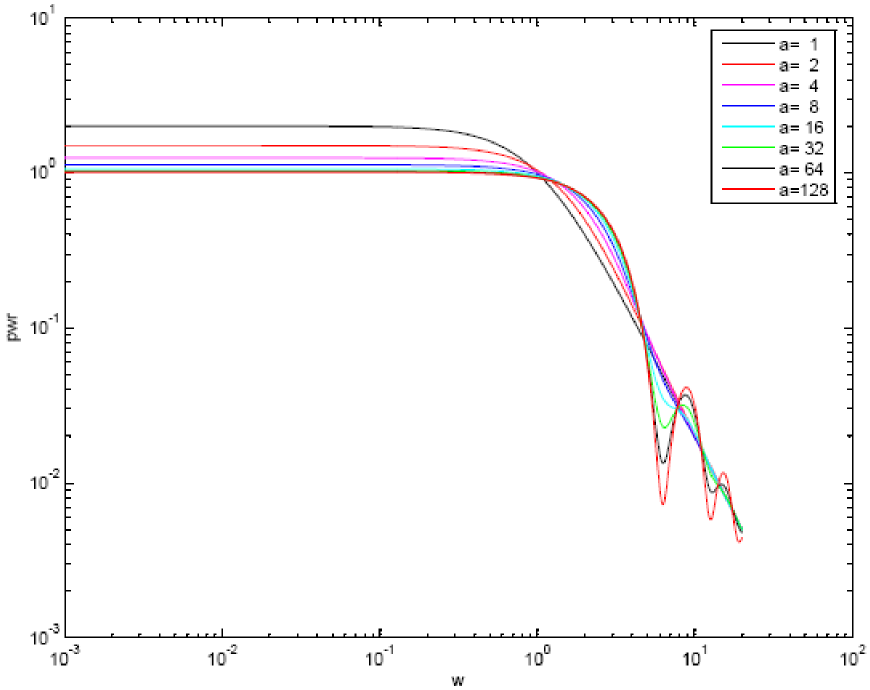
To see the behavior for small *k*, say $r(x,y)$ has variances V_x and V_y, so that

$$\tilde{r}(k_x,k_y)=\exp\left(-\frac{V_xk_x^{\ 2}+V_yk_y^{\ 2}}{2}\right). \tag{5.2.8}$$

For $\left|k\right|^2 V \ll 1$, eq. (5.2.7) becomes,

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega)=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega^2\tilde{\tilde{p}}'(0)}\left(\frac{V_xk_x^{\ 2}+V_yk_y^{\ 2}}{2}\right)+c.c. \tag{5.2.9}$$

For a gamma-process, spec_renewdec_demo.m for a plot of the ω-factor (for $\lambda=1$ and $a=1,2,4,\ldots128$).



For a Poisson process, this reduces to

$$\tilde{\tilde{q}}_{refix-2D}(k_x,k_y,\omega)=\frac{\lambda}{\lambda^2+\omega^2}\left(V_x k_x{}^2+V_y k_y{}^2\right). \tag{5.2.10}$$

Note that the k-term neutralizes a power spectrum of k^{-2} (provided $V_x=V_y$).

Abrupt refixations within a window

Another example is that of saccades with renewal density $p(t)$, but the saccades always land somewhere within a Gaussian window (so they never accumulate a large deviation from the starting position). Let the autocorrelation of this window be $r(x,y)$ with Fourier transform $\tilde{r}(k_x,k_y)$. This is just like the previous case, except that for $N\geq 2$ steps, the decorrelation is no more than for $N=1$.

$$\tilde{\tilde{q}}_{window-2D}(k_x,k_y,\omega)=\tilde{\tilde{p}}_{none}(\omega)+\sum_{1}^{N}\tilde{\tilde{p}}_{first}(\omega)\left(\tilde{\tilde{p}}(\omega)\right)^{N-1}\tilde{\tilde{p}}_{last}(\omega)\tilde{r}(k_x,k_y)+c.c. \tag{5.3.1}$$

or,

$$\tilde{\tilde{q}}_{window-2D}(k_x,k_y,\omega)=\tilde{\tilde{p}}_{none}(\omega)+\frac{\tilde{\tilde{p}}_{first}(\omega)\tilde{\tilde{p}}_{last}(\omega)\tilde{r}(k_x,k_y)}{1-\tilde{\tilde{p}}(\omega)}+c.c. \tag{5.3.2}$$

or,

$$\tilde{\tilde{q}}_{window-2D}(k_x,k_y,\omega)=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega^2\tilde{\tilde{p}}'(0)}-\frac{\left(1-\tilde{\tilde{p}}(\omega)\right)\tilde{r}(k_x,k_y)}{i\omega^2\tilde{\tilde{p}}'(0)}+c.c. \tag{5.3.3}$$

or,

$$\tilde{\tilde{q}}_{window-2D}(k_x,k_y,\omega)=\frac{1-\tilde{\tilde{p}}(\omega)}{i\omega^2\tilde{\tilde{p}}'(0)}\left(1-\tilde{r}(k_x,k_y)\right)+c.c. \tag{5.3.4}$$

For a Poisson process, this reduces to

$$\tilde{\tilde{q}}_{window-2D}(k_x,k_y,\omega)=\frac{2\lambda}{\lambda^2+\omega^2}\left(1-\tilde{r}(k_x,k_y)\right). \tag{5.3.5}$$

For a Gaussian $\tilde{r}(k_x,k_y)$ and $|k|^2 V<<1$, the asymptotic behavior is the same as that of (5.2.7). But note that this case factors exactly; the case in which saccadic movements can accumulate only factors asymptotically.

Constant velocity, Poisson direction-interchanges (one dimension)

This is a model of eye movements that is not intended to be physiologic; instead, it is a highly non-physiologic model, to look at what happens if eye movements are very different from the normal.

I think we can anticipate that this will have the same effect as Brownian motion at sufficiently slow frequencies (but only for orientations that are approximately perpendicular to the movements).

It also might serve as a starting point for analyzing nystagmus.

The model is that eye movements drift with a constant velocity v, but the velocity changes direction randomly (according to a Poisson process), with rate a.

To analyze the process, we note that eye movements evolve in a "state space", in which the state consists of the current direction of the eye movement (*L* or *R*), and, the current displacement, *x*. That is, we can describe the state at time *t* by a pair $\begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix}$, where $h_R(x,t)$ is the probability that the eyes are at a displacement *x* and are moving to the right, and $h_L(x,t)$ is the probability that the eyes are at a displacement *x* and are moving to the left.

We'd like to find eigenvectors for the operator that determines how $\begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix}$ evolves in time. To do this, we start with a Fourier basis, for a reason that will become quickly clear:

$$h_j(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{h}_j(k,t)e^{ikx}dk\,, \tag{5.4.1}$$

for $j=R$ or $j=L$. That is, $\tilde{h}_R(k,t)$ is the projection of the probability distribution onto e^{ikx} of the right-drifting component, and similarly for $\tilde{h}_L(k)$. Note that evolution in time can mix $\tilde{h}_R(k,t)$ and $\tilde{h}_L(k,t)$ at the same spatial frequency *k*, but cannot mix components at different spatial frequencies. So the Fourier transformation determines two-dimensional subspaces in which the probability density evolves, and therefore, these must contain the eigenvectors. Our initial conditions are

$$\begin{pmatrix} h_R(x,0) \\ h_L(x,0) \end{pmatrix}=\frac{1}{2}\begin{pmatrix} \delta(x) \\ \delta(x) \end{pmatrix}, \tag{5.4.2}$$

since the eyes start at the origin, and can be assumed to have equal probability of starting to drift to the left, or to the right.

Eigendecomposition

How does $\begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix}$ evolve over time? Since it is linear, we can analyze it component by component. Say

$$\begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix} = \begin{pmatrix} \tilde{h}_R(k,t) \\ \tilde{h}_L(k,t) \end{pmatrix} e^{ikx}. \tag{5.4.3}$$

In a time step Δt, there is a probability $1-a\Delta t$ that the eye does not change direction, i.e., stays rightward-drifting at velocity *v*. In this case, the rightward density goes from e^{ikx} to $e^{ik(x-v\Delta t)}$, and the leftward density goes from e^{ikx} to $e^{ik(x+v\Delta t)}$. There is a probability $a\Delta t$ that the eye does change direction, and in this case, the density remains e^{ikx}, but the rightward component becomes the leftward component, and vice-versa. Thus,

$$\begin{pmatrix} h_R(x,t+\Delta t) \\ h_L(x,t+\Delta t) \end{pmatrix} = (1-a\Delta t)\begin{pmatrix} \tilde{h}_R(k,t)e^{ikx-ikv\Delta t} \\ \tilde{h}_L(k,t)e^{ikx+ikv\Delta t} \end{pmatrix} + a\Delta t \begin{pmatrix} \tilde{h}_L(k,t) \\ \tilde{h}_R(k,t) \end{pmatrix} e^{ikx}. \tag{5.4.4}$$

For small Δt,

$$\begin{pmatrix} h_R(x,t+\Delta t) \\ h_L(x,t+\Delta t) \end{pmatrix} = (1-a\Delta t)\begin{pmatrix} (1-ikv\Delta t)\tilde{h}_R(k,t) \\ (1+ikv\Delta t)\tilde{h}_L(k,t) \end{pmatrix} e^{ikx} + a\Delta t \begin{pmatrix} \tilde{h}_L(k,t) \\ \tilde{h}_R(k,t) \end{pmatrix} e^{ikx}, \tag{5.4.5}$$

so

$$\frac{d}{dt}\begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix} = \lim_{\Delta t \to 0} \frac{\begin{pmatrix} h_R(x,t+\Delta t) \\ h_L(x,t+\Delta t) \end{pmatrix} - \begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix}}{\Delta t} = \begin{pmatrix} (-a-ikv)\tilde{h}_R(k,t) \\ (-a+ikv)\tilde{h}_L(k,t) \end{pmatrix} e^{ikx} + a\begin{pmatrix} \tilde{h}_L(k,t) \\ \tilde{h}_R(k,t) \end{pmatrix} e^{ikx}, \tag{5.4.6}$$

Making use of eq. (5.4.3),

$$\frac{d}{dt}\begin{pmatrix} \tilde{h}_R(k,t) \\ \tilde{h}_L(k,t) \end{pmatrix} = \begin{pmatrix} -a-ikv & a \\ a & -a+ikv \end{pmatrix} \begin{pmatrix} \tilde{h}_R(k,t) \\ \tilde{h}_L(k,t) \end{pmatrix} = M_k \begin{pmatrix} \tilde{h}_R(k,t) \\ \tilde{h}_L(k,t) \end{pmatrix}, \tag{5.4.7}$$

where

$$M_k = \begin{pmatrix} -a-ikv & a \\ a & -a+ikv \end{pmatrix}. \tag{5.4.8}$$

Eq. (5.4.7) shows how the density evolves in the 2-d space corresponding to the spatial frequency k.

To make this explicit, we find the eigenvalues and eigenvectors of M_k. The eigenvalues of the matrix (5.4.8) are solutions of its characteristic equation,

$$\det|M_k - Iz| = z^2 + 2az + k^2v^2 = 0, \tag{5.4.9}$$

namely,

$$z_{\pm} = -a \pm \sqrt{a^2 - k^2v^2}. \tag{5.4.10}$$

Thus, given any eigenvector $e_k = \begin{pmatrix} b_R \\ b_L \end{pmatrix}$ of eq. (5.4.9), there is a solution of eq. (5.4.7),

$$\begin{pmatrix} \tilde{h}_R(k,t) \\ \tilde{h}_L(k,t) \end{pmatrix} = \begin{pmatrix} e_R \\ e_L \end{pmatrix} e^{zt}. \tag{5.4.11}$$

Without loss of generality, we can assume that the eigenvectors are associated with the eigenvalue z are of the form $\begin{pmatrix} e_R \\ e_L \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix}$. In order for $M_k \begin{pmatrix} 1 \\ x \end{pmatrix} = z \begin{pmatrix} 1 \\ x \end{pmatrix}$, we must have $-a - ikv + ax = z$, from which it follows that the eigenvectors are $\begin{pmatrix} 1 \\ x_+ \end{pmatrix}$ and $\begin{pmatrix} 1 \\ x_- \end{pmatrix}$, with

$$x_\pm = \frac{1}{a}\left(z_\pm + a + ikv\right) = \frac{ikv}{a} \pm \sqrt{1 - k^2v^2/a^2}\ . \tag{5.4.12}$$

At $k = 0$, $z = 0$ or $z = -2a$ and $x_\pm = \pm 1$. At $k = a/v$, $z = -a$ and $x_\pm = i$ (a double root). Note that $x_+x_- = -1$, independent of *k*.

We can superimpose solutions of the form (5.4.11) to obtain a general solution of eq. (5.4.6):

$$\begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(c_+(k) \begin{pmatrix} 1 \\ x_+(k) \end{pmatrix} e^{z_+ t} + c_-(k) \begin{pmatrix} 1 \\ x_-(k) \end{pmatrix} e^{z_- t} \right) e^{ikx} dk\ . \tag{5.4.13}$$

This has the initial condition

$$\begin{pmatrix} h_R(x,0) \\ h_L(x,0) \end{pmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(c_+(k) \begin{pmatrix} 1 \\ x_+(k) \end{pmatrix} + c_-(k) \begin{pmatrix} 1 \\ x_-(k) \end{pmatrix} \right) e^{ikx} dk\ . \tag{5.4.14}$$

Solution

The next step is to express our initial conditions (eq. (5.4.2)) in terms of the eigenvectors. Since $h_R(x,0) = h_L(x,0) = \frac{1}{2}\delta(x)$, we need to find coefficients c_+ and c_- for which

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_+ \begin{pmatrix} 1 \\ x_+ \end{pmatrix} + c_- \begin{pmatrix} 1 \\ x_- \end{pmatrix}, \tag{5.4.15}$$

where $x_\pm$ is given by (5.4.12). It follows that

$$c_+ = \frac{x_- - 1}{2\left(x_- - x_+\right)} \tag{5.4.16}$$

and

$$c_- = \frac{1 - x_+}{2\left(x_- - x_+\right)}. \tag{5.4.17}$$

(where c_+, c_-, x_+, and x_- could depend on k), and these reduce to

$$c_+ = \frac{1}{4}\left(1 + \frac{1 - ikv/a}{\sqrt{1 - k^2v^2/a^2}}\right), \tag{5.4.18}$$

and

$$c_- = \frac{1}{4}\left(1 - \frac{1 - ikv/a}{\sqrt{1 - k^2v^2/a^2}}\right). \tag{5.4.19}$$

Since $x_+x_- = -1$,

$$c_+x_+ = \frac{-1 - x_+}{2\left(x_- - x_+\right)} = \frac{1}{4}\left(1 + \frac{1 + ikv/a}{\sqrt{1 - k^2v^2/a^2}}\right) \tag{5.4.20}$$

and

$$c_- x_- = \frac{1+x_-}{2(x_- - x_+)} = \frac{1}{4}\left(1 - \frac{1+ikv/a}{\sqrt{1-k^2v^2/a^2}}\right). \tag{5.4.21}$$

Eq. (5.4.13) now provides the solution from our initial conditions:

$$\begin{aligned}
\begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix} &= \frac{1}{8\pi}\int_{-\infty}^{\infty}\left(\begin{pmatrix} 1+f(kv/a) \\ 1+f(-kv/a) \end{pmatrix} e^{\left(-a+\sqrt{a^2-k^2v^2}\right)t} + \begin{pmatrix} 1-f(kv/a) \\ 1-f(-kv/a) \end{pmatrix} e^{\left(-a-\sqrt{a^2-k^2v^2}\right)t}\right) e^{ikx} dk \\
&= \frac{e^{-at}}{4\pi}\int_{-\infty}^{\infty}\begin{pmatrix} \cosh\left(\left(\sqrt{a^2-k^2v^2}\right)t\right) + f(kv/a)\sinh\left(\left(\sqrt{a^2-k^2v^2}\right)t\right) \\ \cosh\left(\left(\sqrt{a^2-k^2v^2}\right)t\right) + f(-kv/a)\sinh\left(\left(\sqrt{a^2-k^2v^2}\right)t\right) \end{pmatrix} e^{ikx} dk
\end{aligned}, \tag{5.4.22}$$

where

$$f(u) = \frac{1-iu}{\sqrt{1-u^2}}. \tag{5.4.23}$$

The right and left components (top and bottom) of (5.4.22) should be the same at long times, or, as $a \to \infty$. In the latter case, $u = vk/a \to 0$, $f(u) \to 1$ and both top and bottom of (5.4.22) become

$$h_R(x,t) = h_L(x,t) = \frac{e^{-at}}{4\pi}\int_{-\infty}^{\infty} e^{\left(\sqrt{a^2-v^2k^2}\right)t} e^{ikx} dk = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-\frac{v^2k^2}{2a}t} e^{ikx} dk, \tag{5.4.24}$$

appropriate for Brownian motion provided that v^2 is proportional to a (which makes sense to maintain a diffusion limit).

In the limiting case of $a = 0$, we do not expect that the top and bottom will be the same, because the eye direction never switches. That is, when the eye is moving right (top), it will always be to the right of the origin, at position $x = vt$, and when it is moving left (bottom), it will always be to the left, at position $-vt$. Formally, for $a = 0$, $u = kv/a$ becomes infinite, $f(\infty) \to -1$, $f(-\infty) \to +1$, and (5.4.22) becomes (with $\sinh(iy) = i\sin(iy)$)

$$\begin{aligned}
\begin{pmatrix} h_R(x,t) \\ h_L(x,t) \end{pmatrix} &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\begin{pmatrix} \cos(kvt) - i\sin(kvt) \\ \cos(kvt) + i\sin(kvt) \end{pmatrix} e^{ikx} dk \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\begin{pmatrix} e^{-ikvt} \\ e^{+ikvt} \end{pmatrix} e^{ikx} dk = \frac{1}{4\pi}\int_{-\infty}^{\infty}\begin{pmatrix} e^{ik(x-vt)} \\ e^{ik(x+vt)} \end{pmatrix} dk = \frac{1}{2}\begin{pmatrix} \delta(x-vt) \\ \delta(x+vt) \end{pmatrix}
\end{aligned}, \tag{5.4.25}$$

as expected.

Effect on the input spectrum

To evaluate the effect of eye movements on an image, we are interested in the Fourier transform of the probability of the eye position displacement. This is $\tilde{q}(k,t) = \tilde{h}_R(k,t) + \tilde{h}_L(k,t)$. From (5.4.22),

$$\begin{aligned}
\tilde{q}(k,t) &= e^{-at}\left(\cosh\left(\left(\sqrt{a^2-k^2v^2}\right)t\right) + \frac{1}{\sqrt{1-k^2v^2/a^2}}\sinh\left(\left(\sqrt{a^2-k^2v^2}\right)t\right)\right) \\
&= \frac{1}{2}\left(e^{(-a+\sqrt{a^2-k^2v^2})t} + e^{(-a-\sqrt{a^2-k^2v^2})t} + \frac{1}{\sqrt{1-k^2v^2/a^2}}\left(e^{(-a+\sqrt{a^2-k^2v^2})t} - e^{(-a-\sqrt{a^2-k^2v^2})t}\right)\right)
\end{aligned}, \tag{5.4.26}$$

for $t \geq 0$. For negative values of *t*, it is the same expression, with argument $|t|$. This is a sum of exponentials in *t*. Using eq. (5.1.7),

$$\tilde{\tilde{q}}(k,\omega) = \int_{-\infty}^{\infty} \tilde{q}(k,|t|)e^{-i\omega t}dt = \frac{a-\sqrt{a^2-k^2v^2}}{(a-\sqrt{a^2-k^2v^2})^2+\omega^2} + \frac{a+\sqrt{a^2-k^2v^2}}{(a+\sqrt{a^2-k^2v^2})^2+\omega^2} + \frac{1}{\sqrt{1-k^2v^2/a^2}}\left(\frac{a-\sqrt{a^2-k^2v^2}}{(a-\sqrt{a^2-k^2v^2})^2+\omega^2} - \frac{a+\sqrt{a^2-k^2v^2}}{(a+\sqrt{a^2-k^2v^2})^2+\omega^2}\right). \tag{5.4.27}$$

This simplifies dramatically via straightforward algebra:

$$\tilde{\tilde{q}}(k,\omega) = \frac{1}{\left(\left(a-\sqrt{a^2-k^2v^2}\right)^2+\omega^2\right)\left(\left(a+\sqrt{a^2-k^2v^2}\right)^2+\omega^2\right)} \cdot \left(2a(k^2v^2+\omega^2) + \frac{2\sqrt{a^2-k^2v^2}}{\sqrt{1-k^2v^2/a^2}}\left(k^2v^2-\omega^2\right)\right). \tag{5.4.28}$$

$$\tilde{\tilde{q}}(k,\omega) = \frac{1}{\left(\left(a-\sqrt{a^2-k^2v^2}\right)^2+\omega^2\right)\left(\left(a+\sqrt{a^2-k^2v^2}\right)^2+\omega^2\right)} \cdot \left(2a(k^2v^2+\omega^2) + 2a\left(k^2v^2-\omega^2\right)\right). \tag{5.4.29}$$

$$\tilde{\tilde{q}}(k,\omega) = \frac{4ak^2v^2}{4\omega^2a^2+\left(k^2v^2-\omega^2\right)^2}. \tag{5.4.30}$$

Contact with Brownian case

To make contact with the one-dimensional Brownian case, we consider the large-a limit (rapid switching), and choose

$$v^2 = 2aD. \tag{5.4.31}$$

This is dimensionally correct; v is length/time, D is length^2/time, and a is time^{-1}. Eq. (5.4.30) becomes

$$\tilde{\tilde{q}}(k,\omega) = \frac{8k^2a^2D}{4\omega^2a^2+\left(2k^2aD-\omega^2\right)^2}. \tag{5.4.32}$$

In the regime of $k^2aD >> \omega^2$, this becomes

$$\tilde{\tilde{q}}(k,\omega) = \frac{2k^2D}{\omega^2+k^4D^2} \tag{5.4.33}$$

which matches (5.1.8), and yields the k^2 behavior needed to neutralize the power spectrum of natural scenes for sufficiently high temporal frequencies ω. In the regime of $k^2aD << \omega^2$ or $k^2v^2 << 2\omega^2$ (temporal frequencies faster than the typical reversal), this becomes

$$\tilde{\tilde{q}}(k,\omega) = \frac{8k^2a^2D}{4\omega^2a^2+\omega^4}, \tag{5.4.34}$$

which also yields the k^2 behavior needed to neutralize the power spectrum.

Contact with constant-velocity motion

For steady smooth motion ($a = 0$), eq. (5.4.30) is zero except when $\omega = kv$, as expected: a spatial frequency is translated into a temporal frequency.

Constant-velocity, gamma-process reversal rates

Note that more elaborate "state-space" models can readily be handled. For example, say the distribution of times to reverse direction are goverened by a gamma-process of order $g = 4$. That is, there's a hidden Poisson process of rate *a*, and, after four of these hidden events occur, the eyes reverse direction. In this case, $\tilde{h}_R$ is replaced by $\begin{pmatrix} \tilde{h}_{R,1} \\ \tilde{h}_{R,2} \\ \tilde{h}_{R,3} \\ \tilde{h}_{R,4} \end{pmatrix}$, similarly for $\tilde{h}_L$, and eq. (5.4.8) is replaced by

$$M_4 = \begin{pmatrix} -a-ikv & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a-ikv & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a-ikv & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a-ikv & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a+ikv & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a+ikv & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a+ikv & a \\ a & 0 & 0 & 0 & 0 & 0 & 0 & -a+ikv \end{pmatrix}. \tag{5.5.1}$$

Eigenvalues

To calculate its eigenvalues, we consider $M'_g = aI + M_g$. This is a matrix like

$$M'_g = \begin{pmatrix} -ikv & a & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -ikv & a & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -ikv & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +ikv & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +ikv & a & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ a & 0 & 0 & 0 & 0 & 0 & 0 & +ikv \end{pmatrix}. \tag{5.5.2}$$

All of its invariants except the determinant do not depend on *a*. When $a = 0$, the matrix is diagonal, with g eigenvalues $-ikv$ and g eigenvalues $+ikv$. One can also see (by direct expansion) that

$$\det(M'_g) = (-ikv)^g(+ikv)^g - a^{2g} = k^{2g}v^{2g} - a^{2g} \tag{5.5.3}$$

Therefore,

$$\det(M'_g - yI) = (y+ikv)^g(y-ikv)^g + a^{2g} = (y^2+k^2v^2)^g - a^{2g}. \tag{5.5.4}$$

The eigenvalues of M'_g are roots of $\det(M'_g - yI) = 0$, namely,

$$y_{\pm,j} = \pm\sqrt{a^2\varphi^j - k^2v^2}, \tag{5.5.5}$$

where φ is a nontrivial g th root of unity and $j \in \{0,1,\cdots,g-1\}$, and the eigenvalues of $M_g = -aI + M'_g$ are

$$z_{\pm,j} = -a + y_{\pm,j} = -a \pm \sqrt{a^2\varphi^j - k^2v^2}\ , \tag{5.5.6}$$

generalizing (5.4.10).

Eigenvectors

The initial conditions, generalizing (5.4.2), are

$$\begin{pmatrix} h_{R,1}(x,0) \\ \vdots \\ h_{R,g}(x,0) \\ h_{L,1}(x,0) \\ \vdots \\ h_{L,g}(x,0) \end{pmatrix} = \frac{1}{2g}\begin{pmatrix} \delta(x) \\ \vdots \\ \delta(x) \end{pmatrix}; \tag{5.5.7}$$

we need to express this in terms of the eigenvectors.

As a warm-up, we re-express the results of the $g=1$-analysis of eqs. (5.4.15) to (5.4.21) (writing the initial conditions as a sum of eigenvectors) in a way that better respects the symmetry of the problem. That is, we seek coefficients b_+ and b_- for which

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = b_+\begin{pmatrix} 1 \\ r_+ \end{pmatrix} + b_-\begin{pmatrix} r_- \\ 1 \end{pmatrix}. \tag{5.5.8}$$

It follows that

$$b_+ = c_+ = \frac{1}{4}\left(1 + \frac{1 - ikv/a}{\sqrt{1 - k^2v^2/a^2}}\right), \tag{5.5.9}$$

$$r_+ = x_+ = \frac{ikv}{a} + \sqrt{1 - k^2v^2/a^2}\ , \tag{5.5.10}$$

$$b_- = c_- x_- = \frac{1}{4}\left(1 - \frac{1 + ikv/a}{\sqrt{1 - k^2v^2/a^2}}\right), \tag{5.5.11}$$

and

$$r_- = \frac{1}{x_-} = -x_+ = -\frac{ikv}{a} - \sqrt{1 - k^2v^2/a^2}\ , \tag{5.5.12}$$

and one can verify that $b_- r_- = c_-$.

To find the eigenvectors, we note that for a general matrix of the form

$$M = \begin{pmatrix} p_0 & q_0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ q_{N-1} & 0 & 0 & p_{N-1} \end{pmatrix}, \tag{5.5.13}$$

then an eigenvector $\vec{w} = \begin{pmatrix} w_0 \\ \vdots \\ w_{N-1} \end{pmatrix}$ and eigenvalue z must satisfy

$$w_n z = p_n w_n + q_n w_{n+1} \tag{5.5.14}$$

(with subscripts interpreted cyclically mod N), which means that

$$w_{n+1} = \frac{z - p_n}{q_n} w_n \,. \tag{5.5.15}$$

Here,

$$p_n = \begin{cases} -a - ikv, & n = 0,\ldots,g-1 \\ -a + ikv, & n = g,\ldots,2g-1\,, \end{cases} \tag{5.5.16}$$
$$q_n = a$$

and the eigenvalues z are given by eq. (5.5.6). So,

$$\frac{z_{\pm,j} - p}{q_n} = \begin{cases} \dfrac{ikv}{a} \pm \sqrt{\varphi^j - k^2v^2/a^2}, & n = 0,\ldots,g-1 \\ -\dfrac{ikv}{a} \pm \sqrt{\varphi^j - k^2v^2/a^2}, & n = g,\ldots,2g-1 \end{cases}. \tag{5.5.17}$$

Note that the product of the two alternatives in eq. (5.5.17) is

$$\left(\frac{ikv}{a} \pm \sqrt{\varphi^j - k^2v^2/a^2}\right)\left(-\frac{ikv}{a} \pm \sqrt{\varphi^j - k^2v^2/a^2}\right) = \frac{k^2v^2}{a^2} + \left(\varphi^j - k^2v^2/a^2\right) = \varphi^j \,. \tag{5.5.18}$$

So, with

$$u_j = \frac{ikv}{a} + \sqrt{\varphi^j - k^2v^2/a^2}\,, \tag{5.5.19}$$

we can write a matrix whose columns are the eigenvectors:

$$Z_g = \left(\begin{array}{ccc|ccc} \cdots & 1 & \cdots & \cdots & \left(-u_j\right)^g & \cdots \\ \cdots & u_j & \cdots & \cdots & \varphi^j\left(-u_j\right)^{g-1} & \cdots \\ & \vdots & & & \vdots & \\ \cdots & u_j^{\,g-1} & \cdots & \cdots & \left(\varphi^j\right)^{g-1}\left(-u_j\right) & \cdots \\ \hline \cdots & u_j^{\,g} & \cdots & \cdots & 1 & \cdots \\ \cdots & \varphi^j u_j^{\,g-1} & \cdots & \cdots & -u_j & \cdots \\ \cdots & \vdots & & & \vdots & \\ \cdots & \left(\varphi^j\right)^{g-1} u_j & \cdots & \cdots & \left(-u_j\right)^{g-1} & \cdots \end{array}\right), \tag{5.5.20}$$

where we have taken $w_0 = 1$ for the first set of g columns ($z_{+,j}$ in eq. (5.5.17), $j = 0,\ldots,g-1$), and $w_g = 1$ for the second set of g columns ($z_{-,j}$ in eq. (5.5.17), $j = 0,\ldots,g-1$).

To express the initial conditions in terms of the eigenvectors, we want to solve

$$Z_g \vec{c} = \frac{1}{2g}\vec{1}\,. \tag{5.5.21}$$

The n th element of $\vec{c}$ indicates the weight of the n th eigenvector $\vec{z}_n$ (the n th column of Z_g) corresponding to an eigenvalue z_n, and exponential timecourse $e^{z_n t}$. Here, we use $n \in \{0,...,g-1\}$ as equivalent to the pair $(+, j)$, and $n \in \{g,...,2g-1\}$ as equivalent to the pair $(-, j)$,Determining the contribution to the density requires that we sum over all $2g$ sheets, i.e., that we multiply this element of $\vec{c}$ by the sum of the elements of this eigenvector, namely, computing $\vec{1}^T Z_g$.

To solve (5.5.21), we will make use of the fact that the row eigenvectors of M_g and the column eigenvectors are orthogonal. That is, say R_g is a matrix whose rows are the row eigenvectors of (5.5.20), and that Λ is a diagonal matrix whose diagonal elements list the $2g$ eigenvalues of M_g. Then,

$$R_g Z_g \Lambda = R_g M_g Z_g = \Lambda R_g Z_g . \qquad (5.5.22)$$

This means that $R_g Z_g$ and Λ commute. Provided that the eigenvalues of M_g are distinct (which is generic, see eq. (5.5.6)), then this in turn means that each row eigenvector is orthogonal to all column eigenvectors, except the one with the same eigenvalue.

We can now write (from eq. (5.5.21)):

$$R_g Z_g \vec{c} = \frac{1}{2g} R_g \vec{1} . \qquad (5.5.23)$$

so

$$\vec{c} = \frac{1}{2g} \left(R_g Z_g \right)^{-1} R_g \vec{1} , \qquad (5.5.24)$$

where the matrix inversion is trivial since $R_g Z_g$ is diagonal. Finally, the weight of the exponential corresponding to the n th eigenvalue is the n th component of this expression, multiplied by the sum of the values in the n th column eigenvector, namely, $\vec{1}^T Z_g$. Thus, this coefficient is

$$d_n = \frac{1}{2g} \left(\left(R_g Z_g \right)^{-1} \right)_{n,n} \left(R_g \vec{1} \right)_{n,1} \left(\vec{1}^T Z_g \right)_{1,n} . \qquad (5.5.25)$$

Thus, the next step is to compute the matrix R_g of row eigenvectors. We use the method of eqs. (5.5.13) to (5.5.20). For a general matrix of the form (5.5.13), a row eigenvector $\vec{v}^T = \begin{pmatrix} v_0 & \cdots & v_{N-1} \end{pmatrix}$ with eigenvalue z must satisfy

$$v_n z = p_n v_n + q_n v_{n-1} \qquad (5.5.26)$$

(with subscripts interpreted cyclically mod N), which means that

$$v_{n-1} = \frac{z - p_n}{q_n} v_n . \qquad (5.5.27)$$

p_n and q_n are again given by eq. (5.5.16), and z by eq. (5.5.17). Thus,

$$R_g = \left(\begin{array}{cccc|cccc} \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ u_j^{g-1} & \cdots & u_j & 1 & \left(\varphi^j\right)^{g-1} u_j & \cdots & \varphi^j u_j^{g-1} & u_j^g \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \hline \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \left(\varphi^j\right)^{g-1}\left(-u_j\right) & \cdots & \varphi^j\left(-u_j\right)^{g-1} & \left(-u_j\right)^g & \left(-u_j\right)^{g-1} & \cdots & -u_j & 1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \end{array}\right), \tag{5.5.28}$$

where we have taken $v_{g-1} = 1$ for the first set of g rows ($z_{+,j}$ in eq. (5.5.17), $j = 0,...,g-1$), and $v_{2g-1} = 1$ for the second set of g rows ($z_{-,j}$ in eq. (5.5.17), $j = 0,...,g-1$).

The diagonal elements of $R_g Z_g$ are given by

$$(R_g Z_g)_{j,j} = g\left(u_j^{g-1} + \left(\varphi^j\right)^{g-1} u_j^{g+1}\right) = g u_j^g \left(u_j^{-1} + \left(\varphi^{-j} u_j\right)\right) \tag{5.5.29}$$

and

$$(R_g Z_g)_{g+j,g+j} = g\left(\left(-u_j\right)^{g-1} + \left(\varphi^j\right)^{g-1}\left(-u_j\right)^{g+1}\right) = -g\left(-u_j\right)^g \left(u_j^{-1} + \left(\varphi^{-j} u_j\right)\right) \tag{5.5.30}$$

(with $j \in \{0,...,g-1\}$).

We determine $\vec{1}^T Z_g$ from eq. (5.5.20). The result follows easily by summing geometric series:

$$\vec{1}^T Z_g = \left(\begin{array}{c} \vdots \\ \dfrac{\left(1-u_j^g\right)\left(\varphi^j - u_j^2\right)}{\left(1-u_j\right)\left(\varphi^j - u_j\right)} \\ \vdots \\ \hline \vdots \\ \dfrac{\left(1-\left(-u_j\right)^g\right)\left(\varphi^j - u_j^2\right)}{\left(1+u_j\right)\left(\varphi^j + u_j\right)} \\ \vdots \end{array}\right). \tag{5.5.31}$$

Since the values in a row of R_g are the same as the values in the corresponding column of Z_g (but in reverse order), it follows that

$$R_g \vec{1}^T = \left(\vec{1}^T Z_g\right)^T. \tag{5.5.32}$$

Finally, we can obtain the coefficient of the nth exponential decay mode from eq. (5.5.25), using (5.5.29) through (5.5.32):

$$d_j = \frac{1}{2g^2} \frac{1}{u_j^g \left(u_j^{-1} + \varphi^{-j} u_j\right)} \left(\frac{1-u_j^g}{1-u_j}\right)^2 \left(\frac{\varphi^j - u_j^2}{\varphi^j - u_j}\right)^2. \tag{5.5.33}$$

and

$$d_{g+j} = -\frac{1}{2g^2}\frac{1}{\left(-u_j\right)^g\left(u_j^{-1}+\varphi^{-j}u_j\right)}\left(\frac{1-\left(-u_j\right)^g}{1+u_j}\right)^2\left(\frac{\varphi^j-u_j^{\,2}}{\varphi^j+u_j}\right)^2 \tag{5.5.34}$$

with $j \in \{0,\ldots,g-1\}$.

Note that

$$\varphi^j + u_j^{\,2} = 2u_j\sqrt{\varphi^j - k^2v^2/a^2}\ . \tag{5.5.35}$$

So there's an alternate form for (5.5.33) and (5.5.34):

$$\begin{aligned} d_j &= \frac{1}{2g^2}\frac{u_j\varphi^j}{u_j^{\,g}\left(\varphi^j+u_j^{\,2}\right)}\left(\frac{1-u_j^{\,g}}{1-u_j}\right)^2\left(\frac{\varphi^j-u_j^{\,2}}{\varphi^j-u_j}\right)^2 \\ &= \frac{1}{4g^2}\frac{\varphi^j}{u_j^{\,g}\sqrt{\varphi^j-k^2v^2/a^2}}\left(\frac{1-u_j^{\,g}}{1-u_j}\right)^2\left(\frac{\varphi^j-u_j^{\,2}}{\varphi^j-u_j}\right)^2. \end{aligned} \tag{5.5.36}$$

and

$$d_{g+j} = -\frac{1}{4g^2}\frac{\varphi^j}{\left(-u_j\right)^g\sqrt{\varphi^j-k^2v^2/a^2}}\left(\frac{1-\left(-u_j\right)^g}{1+u_j}\right)^2\left(\frac{\varphi^j-u_j^{\,2}}{\varphi^j+u_j}\right)^2. \tag{5.5.37}$$

To check for $g=1$ (with $j=0$): Eq. (5.5.33) becomes

$$d_0 = \frac{1}{2}\frac{(1+u_0)^2}{u_0^{\,2}+1} = \frac{1}{2}\left(1+\frac{2u_0}{u_0^{\,2}+1}\right) = \frac{1}{2}\left(1+\frac{1}{\sqrt{1-k^2v^2/a^2}}\right); \tag{5.5.38}$$

the coefficient of e^{z_+t} in (5.4.26). For the other eigenvalue, eq. (5.5.33) becomes

$$d_1 = \frac{1}{2}\frac{\left(1-u_0\right)^2}{u_0^{\,2}+1} = \frac{1}{2}\left(1-\frac{2u_0}{u_0^{\,2}+1}\right) = \frac{1}{2}\left(1-\frac{1}{\sqrt{1-k^2v^2/a^2}}\right), \tag{5.5.39}$$

the coefficient of e^{z_-t} both as required for (5.4.26).

Synthesis of the solution

The eigenvalues $z_{\pm,j}$ are related to the u's by (see eqs. (5.5.19) and (5.5.6)):

$$z_{\pm,j} = -a \pm (au_j - ikv), \tag{5.5.40}$$

So

$$\tilde{q}(k,t) = e^{-at}\sum_{j=0}^{g-1}\left(d_j e^{(au_j-ikv)t} + d_{g+j}e^{(-au_j+ikv)t}\right), \tag{5.5.41}$$

$$\tilde{\tilde{q}}(k,\omega) = \int_{-\infty}^{\infty}\tilde{q}(k,|t|)e^{-i\omega t}dt = 2\sum_{j=0}^{g-1}\left(\frac{a-\sqrt{a^2\varphi^j-k^2v^2}}{(a-\sqrt{a^2\varphi^j-k^2v^2})^2+\omega^2}d_j + \frac{a+\sqrt{a^2\varphi^j-k^2v^2}}{(a+\sqrt{a^2\varphi^j-k^2v^2})^2+\omega^2}d_{g+j}\right). \tag{5.5.42}$$

The denominator will clear; the common denominator is

$$
\begin{aligned}
D_j &= \left((a-\sqrt{a^2\varphi^j-k^2v^2})^2+\omega^2\right)\left((a+\sqrt{a^2\varphi^j-k^2v^2})^2+\omega^2\right)= \\
&\omega^4+\omega^2\left((a-\sqrt{a^2\varphi^j-k^2v^2})^2+(a+\sqrt{a^2\varphi^j-k^2v^2})^2\right)+ \\
&(a-\sqrt{a^2\varphi^j-k^2v^2})^2(a+\sqrt{a^2\varphi^j-k^2v^2})^2= \\
&\omega^4+2\omega^2\left(a^2+a^2\varphi^j-k^2v^2\right)+\left(a^2-\left(a^2\varphi^j-k^2v^2\right)\right)^2= \\
&\omega^4+2\omega^2\left(a^2(1+\varphi^j)-k^2v^2\right)+\left(a^2(1-\varphi^j)+k^2v^2\right)^2= \\
&4\omega^2a^2+\left(\omega^2-a^2(1-\varphi^j)-k^2v^2\right)^2
\end{aligned}
\quad . \tag{5.5.43}
$$

Note the resonances that appear for $j \geq 1$, i.e., $g \geq 2$.

The numerator:

$$
\begin{aligned}
N_j &= \left(a-\sqrt{a^2\varphi^j-k^2v^2}\right)\left((a+\sqrt{a^2\varphi^j-k^2v^2})^2+\omega^2\right)d_j+ \\
&\left(a+\sqrt{a^2\varphi^j-k^2v^2}\right)\left((a-\sqrt{a^2\varphi^j-k^2v^2})^2+\omega^2\right)d_{g+j} \\
&=\left(\left(a+\sqrt{a^2\varphi^j-k^2v^2}\right)\left(a^2-a^2\varphi^j+k^2v^2\right)+\omega^2\left(a-\sqrt{a^2\varphi^j-k^2v^2}\right)\right)d_j+ \\
&\left(\left(a-\sqrt{a^2\varphi^j-k^2v^2}\right)\left(a^2-a^2\varphi^j+k^2v^2\right)+\omega^2\left(a+\sqrt{a^2\varphi^j-k^2v^2}\right)\right)d_{g+j} \\
&=a\left(a^2-a^2\varphi^j+k^2v^2+\omega^2\right)(d_j+d_{g+j})+\sqrt{a^2\varphi^j-k^2v^2}\left(a^2-a^2\varphi^j+k^2v^2-\omega^2\right)(d_j-d_{g+j})
\end{aligned}
\quad . \tag{5.5.44}
$$

From (5.5.36) and (5.5.37),

$$
\begin{aligned}
d_j \pm d_{g+j} &= \frac{1}{4g^2}\frac{\varphi^j\left(\varphi^j-u_j^2\right)^2}{u_j^g\sqrt{\varphi^j-k^2v^2/a^2}}\left(\left(\frac{1-u_j^g}{1-u_j}\right)^2\left(\frac{1}{\varphi^j-u_j}\right)^2\pm(-1)^{g+1}\left(\frac{1-(-u_j)^g}{1+u_j}\right)^2\left(\frac{1}{\varphi^j+u_j}\right)^2\right) \\
&=\frac{1}{4g^2}\frac{\varphi^j}{u_j^g\sqrt{\varphi^j-k^2v^2/a^2}}\left(\left(\frac{1-u_j^g}{1-u_j}\right)^2\left(\varphi^j+u_j\right)^2\pm(-1)^{g+1}\left(\frac{1-(-u_j)^g}{1+u_j}\right)^2\left(\varphi^j-u_j\right)^2\right) \\
&=\frac{1}{4g^2}\frac{\varphi^j}{u_j^g\left(1-u_j^2\right)^2\sqrt{\varphi^j-k^2v^2/a^2}}\cdot \\
&\left(\left(1-u_j^g\right)^2\left(1+u_j\right)^2\left(\varphi^j+u_j\right)^2\pm(-1)^{g+1}\left(1-(-u_j)^g\right)^2\left(1-u_j\right)^2\left(\varphi^j-u_j\right)^2\right)
\end{aligned}
\quad . \tag{5.5.45}
$$

For g even:

$$
d_j \pm d_{g+j} = \frac{1}{4g^2}\frac{\varphi^j\left(1-u_j^g\right)^2}{u_j^g\left(1-u_j^2\right)^2\sqrt{\varphi^j-k^2v^2/a^2}}\left(\left(1+u_j\right)^2\left(\varphi^j+u_j\right)^2\mp\left(1-u_j\right)^2\left(\varphi^j-u_j\right)^2\right). \tag{5.5.46}
$$

For g odd:

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Ornstein-Uhlenbeck Process

We consider the one-dimensional case first.

The probability distribution evolves according to

$$\frac{\partial}{\partial t}q(x,t)=\theta\frac{\partial}{\partial x}(xq(x,t))+D_{OUP}\frac{\partial^2}{\partial x^2}q(x,t)\,, \tag{5.6.1}$$

a generalization of eq. (5.1.1). It is well-known that with the initial condition $q(x,0)=\delta(x)$, the solution is

$$q(x,\tau)_{OUP}=\frac{1}{\sqrt{2\pi V(\tau)}}\exp(-\frac{x^2}{2V(\tau)})\,. \tag{5.6.2}$$

where

$$V(\tau)=\frac{D_{OUP}}{\theta}(1-e^{-2\theta\tau})\,. \tag{5.6.3}$$

(One verify this by substituting (5.6.2) into (5.6.1).) For $\tau >> 1/\theta$,

$$V(\tau)=2D_{OUP}\tau\,. \tag{5.6.4}$$

Since (5.6.2) is a Gaussian, standard techniques yield

$$\tilde{q}(k,\tau)_{OUP-1D}=\exp(-\frac{k^2}{2}V(|\tau|))\,, \tag{5.6.5}$$

i.e., V plays the same role as $\langle x^2\rangle = 2D_{Brownian}\tau$.

So,

$$\tilde{\tilde{q}}_{OUP-1D}(k,\omega)=\int_{-\infty}^{\infty}\exp\left(-\frac{k^2D_{OUP}}{2\theta}(1-e^{-2\theta|\tau|})\right)\exp(-i\omega\tau)d\tau \tag{5.6.6}$$

OUP asymptotics, Brownian range

Asymptotics for $\frac{|k|^2 D_{OUP}}{\theta}$ large. Then the approximation that $\left(1-e^{-2\theta|\tau|}\right)\approx 2\theta|\tau|$ is good, since the only contribution to the integral is when the quantity is small. This is the limit that the "pull" of the OUP does not matter very much. Here, the symptotics for the Brownian analysis apply. That is, provided $|k| << \sqrt{\omega / D_{OUP}}$ (i.e., $\omega >> |k|^2 D_{OUP}$) as well as $|k|^2 D_{OUP} >> \theta$, then $\tilde{\tilde{q}}_{OUP-1D}(k,\omega)\approx 2k^2D_{OUP}\omega^{-2}$.

Further analysis when $\frac{|k|^2 D_{OUP}}{\theta}$ is small: Break the integral (5.6.6) into two parts: one part corresponding to the asymptotic behavior when $\theta|\tau|$ is large, and the rest. So

$$\tilde{\tilde{q}}_{OUP-1D}(k,\omega)=\tilde{\tilde{q}}_{dc}(k,\omega)+\tilde{\tilde{q}}_{ac}(k,\omega)\,, \tag{5.6.7}$$

where

$$\tilde{\tilde{q}}_{dc}(k,\omega) = \int_{-\infty}^{\infty} \exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right)\exp(-i\omega\tau)d\tau = 2\pi\delta(\omega)\exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right), \tag{5.6.8}$$

and

$$\tilde{\tilde{q}}_{ac}(k,\omega) = \int_{-\infty}^{\infty}\left(\exp\left(-\frac{k^2 D_{OUP}}{2\theta}(1-e^{-2\theta|\tau|})\right) - \exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right)\right)\exp(-i\omega\tau)d\tau \tag{5.6.9}$$

or

$$\tilde{\tilde{q}}_{ac}(k,\omega) = \exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right)\int_{-\infty}^{\infty}\left(\exp\left(\frac{k^2 D_{OUP}}{2\theta}e^{-2\theta|\tau|}\right)-1\right)\exp(-i\omega\tau)d\tau \tag{5.6.10}$$

Breaking it down into a DC component and an AC component makes sense: the DC component reflects the fact tht because of the Hookes'-law force, the walk never wanders too far from 0. So there's a steady-state component, and this leads to a $\delta(\omega)$-term.

OUP asymptotics, Hooke's law force dominates

For $\frac{|k|^2 D_{OUP}}{\theta}$ small: approximate $\exp\left(-\frac{k^2 D_{OUP}}{2\theta}e^{-2\theta|\tau|}\right)$ by $1-\frac{k^2 D_{OUP}}{2\theta}e^{-2\theta|\tau|}$, so

$$\begin{aligned}
\tilde{\tilde{q}}_{ac}(k,\omega) &\approx \exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right)\int_{-\infty}^{\infty}\frac{k^2 D_{OUP}}{2\theta}e^{-2\theta|\tau|}\exp(-i\omega\tau)d\tau \\
&= \frac{k^2 D_{OUP}}{2\theta}\exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right)\int_{-\infty}^{\infty}e^{-2\theta|\tau|}\exp(-i\omega\tau)d\tau \\
&= \frac{k^2 D_{OUP}}{2\theta}\exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right)\frac{4\theta}{4\theta^2+\omega^2} \\
&= \frac{2k^2 D_{OUP}}{4\theta^2+\omega^2}\exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right)
\end{aligned} \tag{5.6.11}$$

So, for $\frac{|k|^2 D_{OUP}}{\theta}$ small,

$$\tilde{\tilde{q}}_{OUP-1D}(k,\omega) = \tilde{\tilde{q}}_{dc}(k,\omega)+\tilde{\tilde{q}}_{ac}(k,\omega) \approx \left(2\pi\delta(\omega)+\frac{2k^2 D_{OUP}}{4\theta^2+\omega^2}\right)\exp\left(-\frac{k^2 D_{OUP}}{2\theta}\right), \tag{5.6.12}$$

which is not separable. The final factor should be deleted for consistency with "order of smallness":

$$\tilde{\tilde{q}}_{OUP-1D}(k,\omega) \approx \left(2\pi\delta(\omega)+\frac{2k^2 D_{OUP}}{4\theta^2+\omega^2}\right) \tag{5.6.13}$$

For ω that is sufficiently large ($\omega >> 2\theta$), $\tilde{\tilde{q}}_{OUP-1D}(k,\omega) \approx 2k^2 D_{OUP}\omega^{-2}$, even in the large-$\theta$ regime.

OUP asymptotics, full range

At the transition, e.g., for $\frac{|k|^2 D_{OUP}}{\theta}$ near 1 ($k = c\sqrt{\theta / D_{OUP}} = ck_0$, $\omega = a\omega_0 = a\theta$):

$$\tilde{\tilde{q}}_{OUP-1D}(k,\omega)=\tilde{\tilde{q}}_{OUP-1D}(ck_0,a\omega_0)=\int_{-\infty}^{\infty}\exp\left(-c^2\frac{1-e^{-2\theta|\tau|}}{2}\right)\exp(-ia\theta\tau)d\tau \tag{5.6.14}$$

or

$$\begin{aligned}
&\tilde{\tilde{q}}_{OUP-1D}(ck_0,a\omega_0)=\int_{-\infty}^{\infty}\exp\left(-c^2\frac{1-e^{-2\theta|\tau|}}{2}\right)\exp(-ia\theta\tau)d\tau\\
&=\frac{1}{\theta}\int_{-\infty}^{\infty}\exp\left(-c^2\frac{1-e^{-2|u|}}{2}\right)\exp(-iau)du\\
&=\frac{1}{\theta}\int_{-\infty}^{\infty}\left(\exp\left(-c^2\frac{1-e^{-2|u|}}{2}\right)-\exp(-\frac{c^2}{2})\right)\exp(-iau)du+\frac{2\pi\exp(-c^2/2)}{\theta}\delta(a)\\
&=\frac{1}{\theta}f(a,c)+\frac{2\pi\exp(-c^2/2)}{\theta}\delta(a)
\end{aligned} \tag{5.6.15}$$

See eyemov_spec_v2oup_demo.m and figure below, which calculates

$$f(a,c)=\int_{-\infty}^{\infty}\left(\exp\left(-c^2\frac{1-e^{-2|u|}}{2}\right)-\exp(-\frac{c^2}{2})\right)\exp(-iau)du \tag{5.6.16}$$

which, for small c, is given by

$$f(a,c)\approx\frac{2c^2}{4+a^2} \tag{5.6.17}$$

When c is large, the first term in the integrand in is approximated by $1-c^2|u|$ until it gets close to 0. So for large c,

$$f(a,c)\approx\int_{-\infty}^{\infty}\exp\left(-c^2|u|\right)\exp(-iau)du=\frac{2c^2}{c^4+a^2} \tag{5.6.18}$$

A useful approximation over the entire range of c combines eqs. (5.6.17) and (5.6.18):

$$f(a,c)\approx\frac{2c^2}{4+c^4+a^2} \tag{5.6.19}$$

This approximation holds provided that c is either small or large. See illustration below; it is not a bad approximation over the whole range. This leads to a full approximation for the OUP result:

$$\begin{aligned}
&\tilde{\tilde{q}}_{OUP-1D}(k,\omega)=\tilde{\tilde{q}}_{OUP-1D}(ck_0,a\omega_0)\\
&=\frac{1}{\theta}f(a,c)+\frac{2\pi\exp(-c^2/2)}{\theta}\delta(a)\\
&\approx\frac{1}{\theta}\frac{2k^2(D_{OUP}/\theta)}{k^4{D_{OUP}}^2/\theta^2+4+\omega^2/\theta^2}+\frac{2\pi\exp(-k^2D_{OUP}/2\theta)}{\theta}\delta(\frac{\omega}{\theta})\cdot\\
&=\frac{2k^2D_{OUP}}{k^4{D_{OUP}}^2+4\theta^2+\omega^2}+2\pi\exp(-k^2D_{OUP}/2\theta)\delta(\omega)
\end{aligned} \tag{5.6.20}$$

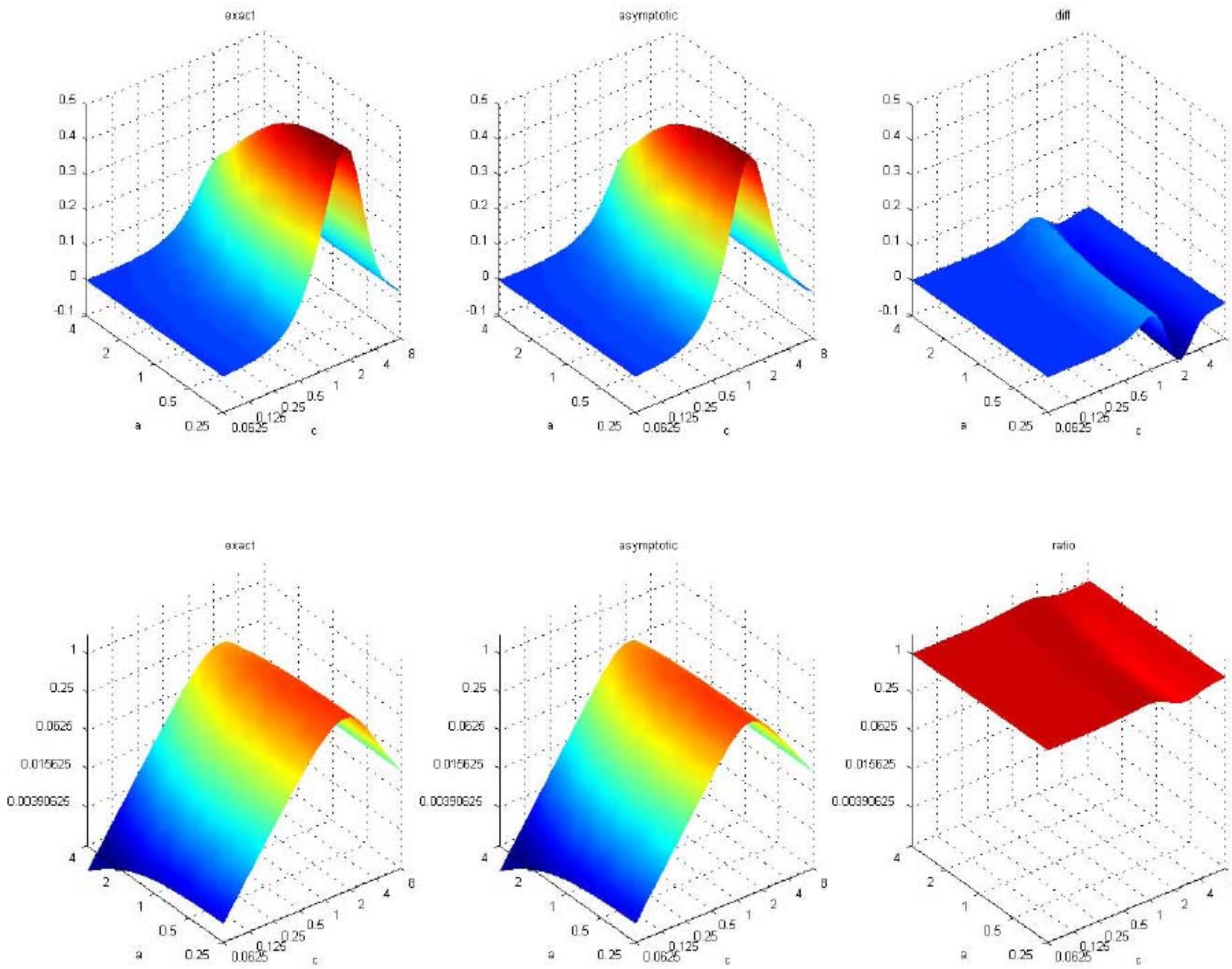
The relationship to the Brownian case is seen with $\theta\to 0$. From the large-c approximation (5.6.18) with $c=k/\sqrt{\theta/D_{OUP}}$, $a=\omega/\theta$, and

$$\tilde{\tilde{q}}_{OUP-1D}(ck_0,a\omega_0)\approx\frac{1}{\theta}f(a,c) \tag{5.6.21}$$

so (using only the large-*c* approximation):

$$\frac{1}{\theta} f(a,c) \approx \frac{1}{\theta} \frac{2k^2 (D_{OUP} / \theta)}{k^4 {D_{OUP}}^2 / \theta^2 + \omega^2 / \theta^2} = \frac{2k^2 D_{OUP}}{k^4 {D_{OUP}}^2 + \omega^2}, \tag{5.6.22}$$

as in eq. (5.1.8), confirming the Brownian limit. This also follows directly from (5.6.20) with $\theta \to 0$.



A more general formulation with saccades as point processes

Here we work out a more general formulation for $q(\Delta x,\tau;t_0)$, the probability that the eyes move by an amount Δx between t_0 and $t=t_0+\tau$, (summed over all starting locations at t_0), given that the last saccade was at time 0.

We assume that the probability of a saccade and the probability distribution of its displacement on the time since the last saccade.

The point process of saccadic occurrences is governed by a renewal process, with renewal density $p(t)$. (More general history-dependences can be handled too.)

The saccades themselves are described by $z(s;\tau)$, which is the probability that a saccade which occurs after an intersaccadic interval τ has a displacement of *s*.

Once the saccade occurs, the fixational eye movement process begins. Fixational eye movements are characterized by $g(\Delta x,\tau;t_0)$, the probability that the eyes move by an amount Δx between t_0 and $t_0+\tau$, given that the last saccade was at time 0.

We calculate $q(\Delta x,\tau;t_0)$ by summing over the number of saccades between t_0 and $t=t_0+\tau$. $q_n(\Delta x,\tau;t_0)$ is the contribution from trajectories with *n* saccades:

$$q_{forward}(\Delta x,\tau;t_0)=\sum_{n=0}^{\infty}q_n(\Delta x,\tau;t_0) \tag{6.1.1}$$

We use $q_{forward}$ rather than *q* to indicate that it only is defined for $\tau\geq 0$ (and can be set to 0 for $\tau<0$).

No saccades

$$q_0(\Delta x,\tau;t_0)=p_{none}(\tau;t_0)g(\Delta x,\tau;t_0), \tag{6.1.2}$$

where $p_{none}(\tau;t_0)$ is the probability that there are no saccades between t_0 and $t=t_0+\tau$, with the last saccade at time 0.

One saccade, at time $t_0+\tau_1$ ($\tau_1<\tau$), assuming that there is a saccade at time 0, and no saccades from then until t_0, and $t_0<\tau_1$:

$$q_1(\Delta x,\tau;t_0)=\int p_{first}(\tau_1;t_0)p_{none}(\tau-\tau_1;0)g(x_1,\tau_1;t_0)z(s_1;t_0+\tau_1)g(\Delta x-x_1-s_1,\tau-\tau_1;0)d\tau_1 ds_1 dx_1, \tag{6.1.3}$$

where $p_{first}(\tau_1;t_0)$ is the probability that the first saccade is at time $\tau_1>t_0$, given that the last saccade at was at time 0.

For (6.1.3), the spatial component is a convolution of the displacement moved before the first saccade (x_1), the displacement moved by the first saccade (s_1), and the displacement moved after the first saccade ($\Delta x-x_1-s_1$). So we can rewrite (6.1.3) as:

$$\tilde{q}_1(k,\tau;t_0)=\int p_{first}(\tau_1;t_0)p_{none}(\tau-\tau_1;0)\tilde{g}(k,\tau_1;t_0)\tilde{z}(k;t_0+\tau_1)\tilde{g}(k,\tau-\tau_1;0)d\tau_1 . \tag{6.1.4}$$

For two saccades, at times $t_0+\tau_1$ and $t_0+\tau_2$ ($\tau_1<\tau_2<\tau$), again assuming that there is a saccade at time 0, and no saccades until t_0:

$$\begin{aligned}&\tilde{q}_2(k,\tau;t_0)=\int p_{first}(\tau_1;t_0)p_{first}(\tau_2-\tau_1;0)p_{none}(\tau-\tau_2;0)\cdot\\&\tilde{g}(k,\tau_1;t_0)\tilde{z}(k;t_0+\tau_1)\tilde{g}(k,\tau_2-\tau_1;0)\tilde{z}(k;\tau_2-\tau_1)\tilde{g}(k,\tau-\tau_2;0)d\tau_1 d\tau_2\end{aligned}, \quad (6.1.5)$$

And in general for *n* saccades, at times $t_0+\tau_i$ ($\tau_1<\tau_2<\ldots<\tau_n<\tau$), again assuming that there is a saccade at time 0, and no saccades until t_0 ($n\geq 1$):

$$\begin{aligned}&\tilde{q}_n(k,\tau;t_0)=\int p_{first}(\tau_1;t_0)\left(\prod_{i=1}^{n-1}p_{first}(\tau_{i+1}-\tau_i;0)\right)p_{none}(\tau-\tau_n;0)\cdot\\&\tilde{z}(k;t_0+\tau_1)\left(\prod_{i=1}^{n-1}\tilde{z}(k;\tau_{i+1}-\tau_i)\right)\cdot\tilde{g}(k,\tau_1;t_0)\left(\prod_{i=1}^{n-1}\tilde{g}(k,\tau_{i+1}-\tau_i;0)\right)\tilde{g}(k,\tau-\tau_n;0)d\tau_1\ldots d\tau_n\end{aligned}, \quad (6.1.6)$$

and for $n=0$,

$$\tilde{q}_0(k,\tau;t_0)=p_{none}(\tau;t_0)\tilde{g}(k,\tau;t_0), \quad (6.1.7)$$

The *p*-term describes the process of saccadic times, the *z*-term describes saccadic lengths, and the g-term describes the fixational eye movements. Note that $\tilde{z}(0,\tau)=\tilde{g}(0,\tau;t)=1$ since $z(\bullet,\tau)$ and $g(\bullet,\tau;t)$ are probability distributions.

To make eq. (6.1.1) into a geometric series, define

$$\tilde{f}(k,\tau;t_0)=p_{first}(\tau;t_0)\tilde{z}(k;t_0+\tau)\tilde{g}(k,\tau;t_0) \quad (6.1.8)$$

and

$$\tilde{f}_{none}(k,\tau;t_0)=p_{none}(\tau;t_0)\tilde{g}(k,\tau;t_0), \quad (6.1.9)$$

Then, eq. (6.1.6) becomes

$$\tilde{q}_n(k,\tau;t_0)=\int\tilde{f}(k,\tau_1;t_0)\left(\prod_{i=1}^{n-1}\tilde{f}(k,\tau_{i+1}-\tau_i;0)\right)\tilde{f}_{none}(k,\tau-\tau_n;0)d\tau_1\ldots d\tau_n, \quad (6.1.10)$$

a convolution in τ, and eq. (6.1.7) becomes

$$\tilde{q}_0(k,\tau;t_0)=\tilde{f}_{none}(k,\tau;t_0). \quad (6.1.11)$$

With the Fourier transforms with respect to τ given by

$$\tilde{\tilde{f}}(k,\omega;t_0)=\int_{-\infty}^{\infty}\tilde{f}(k,\tau;t_0)\exp(-i\omega\tau)d\tau \quad (6.1.12)$$

and

$$\tilde{\tilde{f}}_{none}(k,\omega;t_0)=\int_{-\infty}^{\infty}\tilde{f}_{none}(k,\tau;t_0)\exp(-i\omega\tau)d\tau. \quad (6.1.13)$$

the convolutions become products:

$$\tilde{\tilde{q}}_n(k,\omega;t_0)=\tilde{\tilde{f}}(k,\omega;t_0)\left(\tilde{\tilde{f}}(k,\omega;0)\right)^{n-1}\tilde{\tilde{f}}_{none}(k,\omega;0), \quad (6.1.14)$$

and

$$\tilde{\tilde{q}}_0(k,\omega;t_0)=\tilde{\tilde{f}}_{none}(k,\omega;t_0). \quad (6.1.15)$$

So, the Fourier transform of eq. (6.1.1) becomes

$$\begin{aligned}\tilde{\tilde{q}}_{forward}(k,\omega;t_0) &= \sum_{n=0}^{\infty}\tilde{\tilde{q}}_n(k,\omega;t_0) = \tilde{\tilde{f}}_{none}(k,\omega;t_0) + \sum_{n=1}^{\infty}\tilde{\tilde{f}}(k,\omega;t_0)\left(\tilde{\tilde{f}}(k,\omega;0)\right)^{n-1}\tilde{\tilde{f}}_{none}(k,\omega;0) \\ &= \tilde{\tilde{f}}_{none}(k,\omega;t_0) + \frac{\tilde{\tilde{f}}(k,\omega;t_0)\,\tilde{\tilde{f}}_{none}(k,\omega;0)}{1-\tilde{\tilde{f}}(k,\omega;0)}\end{aligned} \tag{6.1.16}$$

To determine the autocorrelation, we need to do two things: first, consider all possible previous times of the last saccade, and second, consider time intervals τ that are both positive and negative. To do the first, we compute a weighted sum of (6.1.1) over all values of t_0, weighted by the chance that a random time is t_0 after the last saccade, which we denote $p_{prev}(t_0)$. This yields

$$q_{forward}(\Delta x,\tau) = \int_0^{\infty} p_{prev}(t_0)\,q_{forward}(\Delta x,\tau;t_0)\,dt_0\,, \tag{6.1.17}$$

so that

$$\tilde{\tilde{q}}_{forward}(k,\omega) = \int_0^{\infty} p_{prev}(t_0)\,\tilde{\tilde{q}}_{forward}(k,\omega;t_0)\,dt_0\,. \tag{6.1.18}$$

If the quantities that make up $q_{forward}(\Delta x,\tau;t_0)$ are independent of t_0, the above step is trivial, as p_{prev} is unit-normalized. Otherwise, the above can be done by Fourier transform, as it is essentially a convolution in t_0. To calculate $p_{prev}(t_0)$, we use the same logic as used for $p_{first}(t)$ (above, see (3.1.15)). That is, $p_{prev}(t_0)$ contains a contribution from all intervals of length $\tau' \ge t_0$. It is the probability that time t_0 is inside the an interval from a saccade at some time t to the next saccade, at time $t+\tau'$, times the probability that, conditional on being in this interval, that it is $t_0-\tau$ from the end. So it is exactly equal to $p_{first}(t_0)$, and, according to eq. (3.1.16),

$$\tilde{\tilde{p}}_{prev}(\omega) = \frac{1-\tilde{\tilde{p}}(\omega)}{-\omega\tilde{\tilde{p}}'(0)}\,. \tag{6.1.19}$$

Finally, since

$$q(\Delta x,\tau) = q_{forward}(\Delta x,|\tau|) = q_{forward}(\Delta x,\tau) + q_{forward}(\Delta x,-\tau)\,, \tag{6.1.20}$$

we obtain the desired quantity:

$$\tilde{\tilde{q}}(k,\omega) = \tilde{\tilde{q}}_{forward}(k,\omega) + \tilde{\tilde{q}}_{forward}(k,-\omega) = 2\,\mathrm{Re}\left(\tilde{\tilde{q}}_{forward}(k,\omega)\right). \tag{6.1.21}$$

Some special cases

Simplified history dependence

In the case that the quantities $\tilde{\tilde{f}}(k,\omega;t_0)$ and $\tilde{\tilde{f}}_{none}(k,\omega;t_0)$ that make up $q_{forward}(\Delta x,\tau;t_0)$ are independent of t_0, we have (from eq. (6.1.16)):

$$\tilde{\tilde{q}}_{forward}(k,\omega)=\tilde{\tilde{q}}_{forward}(k,\omega;t_0)=\frac{\tilde{\tilde{f}}_{none}(k,\omega)}{1-\tilde{\tilde{f}}(k,\omega)}. \qquad (6.2.1)$$

Very large saccades

A limiting case is that saccades are so large so that they effectively move to a new image, uncorrelated, image. Thus, the probability $z(s,t)$ of a saccade displacement integrates to 1 (across s), but is arbitrarily small at any nonzero spatial frequency. Since $\tilde{g}(0,\tau;t)=1$ ($g(\bullet,\tau;t)$ is a probability distribution),

$$\tilde{f}(k,\tau;t_0)=\begin{cases} p_{first}(\tau;t_0) & ,k=0 \\ 0 & ,k\neq 0 \end{cases}. \qquad (6.2.2)$$

Poisson saccade occurrence

Saccades governed by Poisson process with rate λ_S, and mean-squared distance (two-dimensional) is $R_S{}^2$. So,

$$p_{none}(\tau;t_0)=e^{-\lambda_S\tau}, \qquad (6.3.1)$$

$$p_{first}(\tau;t_0)=\lambda_S e^{-\lambda_S\tau}, \qquad (6.3.2)$$

and

$$\tilde{z}(k;\tau)=e^{-|k|^2 R_S{}^2/4}. \qquad (6.3.3)$$

where we can either (i) take $R_S{}^2$ a constant, or, (ii) $R_S{}^2=2D_S\tau$, for a “diffusive” variant of eye movements. Since everything is Poisson, there’s no dependence on t_0.

For fixational eye movements, also characterized by a two-dimensional Brownian process,

$$\tilde{g}(k,\tau;t_0)=e^{-|k|^2 D|\tau|/2}. \qquad (6.3.4)$$

So,

$$\tilde{f}_{indep}(k,\tau;t_0)=\lambda_S e^{-\lambda_S\tau}e^{-|k|^2 R_S{}^2/4}e^{-|k|^2 D\tau/2}=\lambda_S\exp\left(-\left(\lambda_S+|k|^2\frac{D}{2}\right)\tau-\frac{|k|^2 R_S{}^2}{4}\right) \qquad (6.3.5)$$

(case (i), saccades don’t depend on previous interval) or

$$\tilde{f}_{diffusive}(k,\tau;t_0)=\lambda_S e^{-\lambda_S\tau}e^{-|k|^2 D_S\tau/2}e^{-|k|^2 D\tau/2}=\lambda_S\exp\left(-\left(\lambda_S+|k|^2\frac{D+D_S}{2}\right)\tau\right). \qquad (6.3.6)$$

(case (ii), “diffusive” dependence of saccade length on previous intersaccadic interval). For either assumption about saccades,

$$\tilde{f}_{none}(k,\tau;t_0)=e^{-\lambda_S\tau}e^{-|k|^2 D\tau/2}=\exp\left(-\left(\lambda_S+|k|^2\frac{D}{2}\right)\tau\right). \qquad (6.3.7)$$

The next step is Fourier transformation with respect to τ. We only consider $\tau\geq 0$.

$$\tilde{\tilde{f}}_{indep}(k,\omega;t_0)=\lambda_S\exp\left(-\frac{|k|^2 R_S{}^2}{4}\right)\frac{1}{\lambda_S+|k|^2\frac{D}{2}+i\omega}=\exp\left(-\frac{|k|^2 R_S{}^2}{4}\right)\frac{1}{1+\left(|k|^2\frac{D}{2}+i\omega\right)/\lambda_S}, \qquad (6.3.8)$$

$$\tilde{\tilde{f}}_{diffusive}(k,\omega;t_0)=\lambda_S\frac{1}{\lambda_S+|k|^2\frac{D+D_S}{2}+i\omega}=\frac{1}{1+\left(|k|^2\frac{D+D_S}{2}+i\omega\right)/\lambda_S}, \qquad (6.3.9)$$

and

$$\tilde{\tilde{f}}_{none}(k,\omega;t_0)=\frac{1}{\lambda_S+|k|^2\frac{D}{2}+i\omega}. \qquad (6.3.10)$$

The simpler case is the diffusive one. With (6.2.1),

$$\tilde{\tilde{q}}_{forward,diffusive}(k,\omega)=\frac{\tilde{\tilde{f}}_{none}(k,\omega)}{1-\tilde{\tilde{f}}_{diffusive}(k,\omega)}=\left(\frac{1}{\lambda_S+|k|^2\frac{D}{2}+i\omega}\right)\left(1+\frac{1}{\left(|k|^2\frac{D+D_S}{2}+i\omega\right)/\lambda_S}\right) \qquad (6.3.11)$$

For saccades that are independent of the preceding intersaccadic interval, (6.2.1) yields

$$\tilde{\tilde{q}}_{forward,indep}(k,\omega)=\frac{\tilde{\tilde{f}}_{none}(k,\omega)}{1-\tilde{\tilde{f}}_{indep}(k,\omega)}=\left(\frac{1}{\lambda_S+|k|^2\frac{D}{2}+i\omega}\right)\left(\frac{1+\left(|k|^2\frac{D}{2}+i\omega\right)/\lambda_S}{1+\left(|k|^2\frac{D}{2}+i\omega\right)/\lambda_S-\exp\left(-\frac{|k|^2 R_S{}^2}{4}\right)}\right). \qquad (6.3.12)$$

As a check: for $\lambda_S=0$ (no saccades), both (6.3.11) and (6.3.12) become simply

$$\tilde{\tilde{q}}_{forward,nosacc}(k,\omega)=\frac{1}{|k|^2\frac{D}{2}+i\omega}. \qquad (6.3.13)$$

For $D_S=0$ or $R_S=0$ (ineffective saccades) both (6.3.11) and (6.3.12) also become

$$\tilde{\tilde{q}}_{forward,nullsacc}(k,\omega)=\left(\frac{1}{\lambda_S+|k|^2\frac{D}{2}+i\omega}\right)\left(1+\frac{1}{\left(|k|^2\frac{D}{2}+i\omega\right)/\lambda_S}\right)$$
$$=\left(\frac{1}{\lambda_S+|k|^2\frac{D}{2}+i\omega}\right)\left(\frac{\left(|k|^2\frac{D}{2}+i\omega\right)/\lambda+1}{\left(|k|^2\frac{D}{2}+i\omega\right)/\lambda_S}\right)=\frac{1}{|k|^2\frac{D}{2}+i\omega}. \qquad (6.3.14)$$

Applying (6.1.21) to either (6.3.13) or (6.3.14),

$$\tilde{\tilde{q}}_{nosacc}(k,\omega)=\tilde{\tilde{q}}_{nullsacc}(k,\omega)=2\,\mathrm{Re}\frac{1}{|k|^2\frac{D}{2}+i\omega}=\frac{|k|^2 D}{|k|^4 D^2/4+\omega^2}, \qquad (6.3.15)$$

in agreement with the result for 2-D Brownian fixational eye movements (eq. (5.1.19)).

Focusing on single saccades, not as point processes

Here we focus on single saccades, not as point processes – perhaps to model the Mostofi et al. work about the transformations due to saccades of different lengths. Possible interest is, 1D vs 2D; the relevant details of the velocity profile, and, understanding the "main sequence" relationship between saccade duration and velocity.

Ramp velocity profile, one dimension

Along with Mostofi et al., we place the saccade in the middle of a time window of length T_{win}, running from $-T_{win}/2$ to $T_{win}/2$. The saccade has a duration of T_{sac} (where $T_{sac} < T_{win}$) and fixed velocity is v_{sac}, and therefore an amplitude $h_{sac} = v_{sac}T_{sac}$. What is $q(\Delta x, \tau; t_0)$, the probability that the eyes move by an amount Δx between t_0 and $t = t_0 + \tau$? Set up three intervals: $I_- = (-\infty, -T_{sac}/2]$, $I_0 = [-T_{sac}/2, T_{sac}/2]$, and $I_+ = [T_{sac}/2, \infty)$.

We will assume that velocity is uniform, and given by the main sequence – which, according to https://www.liverpool.ac.uk/~pcknox/teaching/Eymovs/params.htm, is given by ., duration (ms)=2.2*amplitude(deg)+21, i.e.,

$$T_{sac} = MainSeqSlope * v_{sac}T_{sac} + MainSeqIntercept, \tag{6.4.1}$$

where $MainSeqSlope = 0.0022$ s/deg and $MainSeqIntercept = 0.021$ s. So, $T_{sac} = 0.0022T_{sac}v_{sac} + 0.021$, i.e., $v_{sac} = \dfrac{1 - 0.021/T_{sac}}{0.0022}$, with typical T_{sac} ranging from 0.0232 sec (1 deg) to 0.043 sec (10 deg).

First we consider the one-dimensional case; the saccade can go in either direction along the horizontal. Then we average over all orientations with respect to the horizontal. Displacement as a function of time is

$$s_{ramp}(t) = v_{sac}\left(\min(\max(t + T_{sac}/2, 0), T_{sac}\right). \tag{6.4.2}$$

Then

$$q_{1D}(\Delta x, \tau; t_0) = \frac{1}{2}\left(\delta\left(\Delta x - s_{ramp}(t_0+\tau) + s_{ramp}(t_0)\right) + \delta\left(\Delta x + s_{ramp}(t_0+\tau) - s_{ramp}(t_0)\right)\right). \tag{6.4.3}$$

Our goal is to calculate

$$\begin{aligned}\tilde{\tilde{q}}_{1D}(k,\omega;t_0) &= \iint q_{1D}(\Delta x,\tau;t_0)\exp(-ik\Delta x)\exp(-i\omega\tau)d\Delta x d\tau \\ &= \int_{-\infty}^{\infty} \tilde{q}_{1D}(k,\tau;t_0)\exp(-i\omega\tau)d\tau\end{aligned} \tag{6.4.4}$$

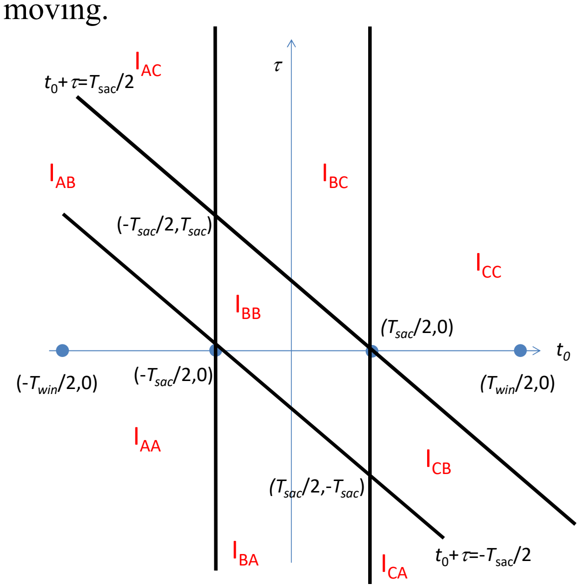
where

$$\tilde{q}_{1D}(k,\tau;t_0) = \int_{-\infty}^{\infty} q_{1D}(\Delta x,\tau;t_0)e^{-ik\Delta x}d\Delta x, \tag{6.4.5}$$

and to take a sensible average over t_0, and take a sensible limit as the range of values increases without bound.

Approach 1. Explicit start time.

This might be the right way to go for detailed comparison with experimental data, but it is messy. Three are many cases, with cutpoints depending on t_0 and $t_0+\tau$. First subscript of *I* refers to beginning of interval, second subscript refers to end. *A* means hasn't yet moved, *B* means moving, *C* means no longer moving.



I_{A*}: If $t_0 \le -T_{sac}/2$, then $s_{ramp}(t_0)=0$.

I_{B*}: If $-T_{sac}/2 \le t_0 \le T_{sac}/2$, then $s_{ramp}(t_0)=v_{sac}(t_0+T_{sac}/2)$.

I_{C*}: If $T_{sac}/2 \le t_0$, then $s_{ramp}(t_0)=v_{sac}T_{sac}$.

I_{*A}: If $t_0+\tau \le -T_{sac}/2$, then $s_{ramp}(t_0+\tau)=0$.

I_{*B}: If $-T_{sac}/2 \le t_0+\tau \le T_{sac}/2$, then $s_{ramp}(t_0+\tau)=v_{sac}(t_0+\tau+T_{sac}/2)$.

I_{*C}: If $T_{sac}/2 \le t_0+\tau$, then $s_{ramp}(t_0+\tau)=v_{sac}T_{sac}$.

So,

I_{AA}: $q(\Delta x,\tau;t_0)=\delta(\Delta x)$; $\tilde{q}(k,\tau;t_0)=1$

I_{AB}: $q(\Delta x,\tau;t_0)=\frac{1}{2}\left(\delta\left(\Delta x-v_{sac}(t_0+\tau+T_{sac}/2)\right)+\delta\left(\Delta x+v_{sac}(t_0+\tau+T_{sac}/2)\right)\right)$;

$\tilde{q}(k,\tau;t_0)=\cos\left(kv_{sac}(t_0+\tau+T_{sac}/2)\right)$

I_{AC}: $q(\Delta x,\tau;t_0)=\frac{1}{2}\left(\delta\left(\Delta x-v_{sac}T_{sac}\right)+\delta\left(\Delta x+v_{sac}T_{sac}\right)\right)$; $\tilde{q}(k,\tau;t_0)=\cos\left(kv_{sac}T_{sac}\right)$

I_{BA}: $q(\Delta x,\tau;t_0)=\frac{1}{2}\left(\delta\left(\Delta x-v_{sac}(t_0+T_{sac}/2)\right)+\delta\left(\Delta x+v_{sac}(t_0+T_{sac}/2)\right)\right)$; $\tilde{q}(k,\tau;t_0)=\cos\left(kv_{sac}(t_0+T_{sac}/2)\right)$

I_{BB}: $q(\Delta x,\tau;t_0)=\frac{1}{2}\left(\delta\left(\Delta x-v_{sac}\tau\right)+\delta\left(\Delta x+v_{sac}\tau\right)\right)$; $\tilde{q}(k,\tau;t_0)=\cos\left(kv_{sac}\tau\right)$

I_{BC}: $q(\Delta x,\tau;t_0)=\frac{1}{2}\left(\delta\left(\Delta x-v_{sac}(t_0-T_{sac}/2)\right)+\delta\left(\Delta x+v_{sac}(t_0-T_{sac}/2)\right)\right)$;

$\tilde{q}(k,\tau;t_0)=\cos\left(kv_{sac}(t_0-T_{sac}/2)\right)$

I_{CA}: $q(\Delta x,\tau;t_0)=\frac{1}{2}\left(\delta\left(\Delta x-v_{sac}T_{sac}\right)+\delta\left(\Delta x+v_{sac}T_{sac}\right)\right)$; $\tilde{q}(k,\tau;t_0)=\cos\left(kv_{sac}T_{sac}\right)$

I_{CB}: $q(\Delta x,\tau;t_0)=\frac{1}{2}\left(\delta\left(\Delta x - v_{sac}(t_0+\tau-T_{sac}/2)\right)+\delta\left(\Delta x + v_{sac}(t_0+\tau-T_{sac}/2)\right)\right)$;
$\tilde{q}(k,\tau;t_0)=\cos\left(kv_{sac}(t_0+\tau-T_{sac}/2)\right)$
I_{CC}: $q(\Delta x,\tau;t_0)=\delta\left(\Delta x\right)$; $\tilde{q}(k,\tau;t_0)=1$

In tabular form:

$\tilde{q}(k,\tau;t_0)$	$value\,of\ t_0$	I_{A*}	I_{B*}	I_{C*}
$value\,of\ t_0+\tau$		$(-\infty,\frac{-T_{sac}}{2}]$	$[-\frac{T_{sac}}{2},\frac{T_{sac}}{2}]$	$[\frac{T_{sac}}{2},\infty)$
I_{*A}	$(-\infty,\frac{-T_{sac}}{2}]$	1	$\cos\left(kv_{sac}(t_0+\frac{T_{sac}}{2})\right)$	$\cos\left(kv_{sac}T_{sac}\right)$
I_{*B}	$[-\frac{T_{sac}}{2},\frac{T_{sac}}{2}]$	$\cos\left(kv_{sac}(t_0+\tau+\frac{T_{sac}}{2})\right)$	$\cos\left(kv_{sac}\tau\right)$	$\cos\left(kv_{sac}(t_0+\tau-\frac{T_{sac}}{2})\right)$
I_{*C}	$[\frac{T_{sac}}{2},\infty)$	$\cos\left(kv_{sac}T_{sac}\right)$	$\cos\left(kv_{sac}(t_0-\frac{T_{sac}}{2})\right)$	1

(6.4.6)

Integrating over τ breaks into three cases, depending on relation of t_0 to $\pm T_{sac}/2$. We assume that we have a finite window of data, from $[-T_{win}/2,T_{win}/2]$. That is, both t_0 and $t_0+\tau$ must be within $[-T_{win}/2,T_{win}/2]$.

1. For $t_0\in[-T_{win}/2,-T_{sac}/2]$,

$$\tilde{\tilde{q}}(k,\omega;t_0)=\int_{-\infty}^{\infty}\tilde{q}(k,\tau;t_0)\exp(-i\omega\tau)d\tau = contribs\ from\, I_{AA}+I_{AB}+I_{AC}$$
$$=\int_{-T_{win}/2-t_0}^{-T_{sac}/2-t_0}\exp(-i\omega\tau)d\tau+\int_{-T_{sac}/2-t_0}^{T_{sac}/2-t_0}\cos\left(kv_{sac}(t_0+\tau+\frac{T_{sac}}{2})\right)\exp(-i\omega\tau)d\tau+\int_{T_{sac}/2-t_0}^{T_{win}/2-t_0}\cos\left(kv_{sac}T_{sac}\right)\exp(-i\omega\tau)d\tau$$
$$=\frac{e^{i\omega t_0}}{i\omega}\left(e^{i\omega\frac{T_{win}}{2}}-e^{i\omega\frac{T_{sac}}{2}}\right)$$
$$+\frac{1}{2}\left\{\frac{e^{i(t_0+\frac{T_{sac}}{2})kv_{sac}}}{i(\omega-kv_{sac})}\left(e^{i(kv_{sac}-\omega)(-T_{sac}/2-t_0)}-e^{i(kv_{sac}-\omega)(T_{sac}/2-t_0)}\right)+\frac{e^{-i(t_0+\frac{T_{sac}}{2})kv_{sac}}}{i(\omega+kv_{sac})}\left(e^{i(kv_{sac}+\omega)(T_{sac}/2+t_0)}-e^{i(kv_{sac}+\omega)(-T_{sac}/2+t_0)}\right)\right\}$$
$$+\frac{e^{i\omega t_0}}{i\omega}\left(e^{-i\omega\frac{T_{sac}}{2}}-e^{-i\omega\frac{T_{win}}{2}}\right)\cos\left(kv_{sac}T_{sac}\right)$$
$$=\frac{e^{i\omega t_0}}{i\omega}\left(e^{i\omega\frac{T_{win}}{2}}-e^{i\omega\frac{T_{sac}}{2}}\right)+\frac{e^{i\omega t_0}}{i\omega}\left(e^{-i\omega\frac{T_{sac}}{2}}-e^{-i\omega\frac{T_{win}}{2}}\right)\cos\left(kv_{sac}T_{sac}\right)$$
$$+\frac{e^{i\omega t_0}}{2}\left\{\frac{1}{i(\omega-kv_{sac})}\left(e^{i\omega T_{sac}/2}-e^{i(-\omega T_{sac}/2+kv_{sac}T_{sac})}\right)+\frac{1}{i(\omega+kv_{sac})}\left(e^{i\omega T_{sac}/2}-e^{i(-\omega T_{sac}/2-kv_{sac}T_{sac})}\right)\right\} \qquad (6.4.7)$$

Now have to work out the other intervals for t_0.

Approach 2. No explicit start time.

Here the motivation is that have to average over t_0 anyway. Assume we are analyzing a large interval of length T_{win} that contains only one saccade. As the length of this interval increases, we can neglect edge effects. The probability that two points separated by τ are affected by a saccade (i.e., that at least part of the saccade occurs in the interval $[t_0, t_0+\tau]$) is proportional to $T_{sac}+|\tau|$, i.e., some $B(T_{sac}+|\tau|)$, where B is a normalization constant. The probability that the total movement in this interval is $v_{sac}T_{sac}$, the saccade length, is proportional to the probability that the saccade lies entirely within the interval, i.e., $B\max(0,|\tau|-T_{sac})$. The probability that the total movement is less than the saccade length is proportional to the difference, namely, $B(T_{sac}+|\tau|-\max(0,|\tau|-T_{sac}))=B(2T_{sac}+|\tau|-\max(T_{sac},|\tau|))$, i.e., either $B(T_{sac}+|\tau|)$ if $T_{sac}\geq|\tau|$ or $2BT_{sac}$ if $T_{sac}\leq|\tau|$, i.e., $T_{sac}+\min(T_{sac},|\tau|)$. The maximum that can be moved in the interval is $v_{sac}\min(T_{sac},|\tau|)$, and the time moved can be anywhere from 0 to $\min(T_{sac},\tau)$, and all values in this interval have equal probability.

So, for displacements Δx up to $v_{sac}\min(T_{sac},|\tau|)$ (but for Δx on both sides of 0), the probability of a movement by this amount is

$$q_{1D}(\Delta x,\tau)dx = B\frac{T_{sac}+\min(T_{sac},|\tau|)}{2v_{sac}\min(T_{sac},|\tau|)}dx, \tag{6.4.8}$$

and for the maximal displacement $|\Delta x| = v_{sac}\min(T_{sac},|\tau|)$, there is an additional point mass component,

$$, \tag{6.4.9}$$

as the total probability must integrate to $B(T_{sac}+|\tau|)$. Note that, properly, the point mass component is only present if $T_{sac}\leq|\tau|$. In both cases, for $\Delta x = 0$, there's also a point-mass component of size $1-B\left(T_{sac}+|\tau|\right)$, so that $\int q_{1D}(\Delta x,\tau)d\Delta x = 1$.

Thus, for $|\tau|\leq T_{sac}$,

$$
\begin{aligned}
\tilde{q}_{1D}(k,\tau) &= \int_{-\infty}^{\infty} q(\Delta x,\tau) e^{-ik\Delta x} d\Delta x \\
&= \int_{-\infty}^{\infty} \left(1 - B\left(T_{sac} + |\tau|\right)\right)\delta(\Delta x) e^{-ik\Delta x} d\Delta x + \int_{-v_{sac}\min(T_{sac},|\tau|)}^{+v_{sac}\min(T_{sac},|\tau|)} \left(B\frac{T_{sac}+\min(T_{sac},|\tau|)}{2v_{sac}\min(T_{sac},|\tau|)}\right) e^{-ik\Delta x} d\Delta x \\
&= 1 - B\left(T_{sac} + |\tau|\right) + B\frac{T_{sac}+\min(T_{sac},|\tau|)}{2v_{sac}\min(T_{sac},|\tau|)} \left(\frac{e^{-ikv_{sac}\min(T_{sac},|\tau|)} - e^{ikv_{sac}\min(T_{sac},|\tau|)}}{-ik} \right) \\
&= 1 - B\left(T_{sac} + |\tau|\right) + B\left(T_{sac} + \min(T_{sac},|\tau|)\right) \left(\frac{\sin(kv_{sac}\min(T_{sac},|\tau|))}{kv_{sac}\min(T_{sac},|\tau|)} \right) \\
&= 1 - B\left(T_{sac} + |\tau|\right) + B\left(T_{sac} + |\tau|\right)\operatorname{sinc}(kv_{sac}|\tau|) \\
&= 1 - B\left(T_{sac} + |\tau|\right)\left(1 - \operatorname{sinc}(kv_{sac}|\tau|)\right)
\end{aligned} \quad , \qquad (6.4.10)
$$

and for $|\tau| \geq T_{sac}$,

$$
\begin{aligned}
\tilde{q}_{1D}(k,\tau) &= 1 - B\left(T_{sac} + |\tau|\right) + B\left(T_{sac} + \min(T_{sac},|\tau|)\right) \left(\frac{\sin(kv_{sac}\min(T_{sac},|\tau|))}{kv_{sac}\min(T_{sac},|\tau|)} \right) + \\
& B\frac{|\tau| - \min(T_{sac},|\tau|)}{2} \int \delta(|\Delta x| - v_{sac}\min(T_{sac},|\tau|)) e^{-ik\Delta x} d\Delta x \\
&= 1 - B\left(T_{sac} + |\tau|\right) + B\left(T_{sac} + \min(T_{sac},|\tau|)\right) \left(\frac{\sin(kv_{sac}\min(T_{sac},|\tau|))}{kv_{sac}\min(T_{sac},|\tau|)} \right) + \\
& B\left(|\tau| - \min(T_{sac},|\tau|)\right)\cos\left(kv_{sac}\min(T_{sac},|\tau|)\right) \\
&= 1 - B\left(T_{sac} + |\tau|\right) + 2BT_{sac}\operatorname{sinc}\left(kv_{sac}T_{sac}\right) + B\left(|\tau| - T_{sac}\right)\cos\left(kv_{sac}T_{sac}\right)
\end{aligned} \quad , \qquad (6.4.11)
$$

Saccades can occur in any direction. Say the angle w.r.t. the horizontal is θ. It suffices to consider $\theta \in [0, \pi/2]$. After projection on the horizontal axis, the effective velocity is $v_{sac}\cos\theta$. Therefore, for $|\tau| \leq T_{sac}$ and $k = |\vec{k}|$,

$$
\tilde{q}_{2D}(\vec{k},\tau) = 1 - B\left(T_{sac} + |\tau|\right) + B\left(T_{sac} + |\tau|\right)\frac{2}{\pi}\int_0^{\pi/2} \operatorname{sinc}(kv_{sac}|\tau|\cos\theta) d\theta , \qquad (6.4.12)
$$

and for $|\tau| \geq T_{sac}$,

$$
\begin{aligned}
\tilde{q}_{2D}(k,\tau) &= 1 - B\left(T_{sac} + |\tau|\right) + 2BT_{sac}\frac{2}{\pi}\int_0^{\pi/2} \operatorname{sinc}\left(kv_{sac}T_{sac}\cos\theta\right) d\theta + B\left(|\tau| - T_{sac}\right)\frac{2}{\pi}\int_0^{\pi/2} \cos\left(kv_{sac}T_{sac}\cos\theta\right) d\theta \\
&= 1 - B\left(T_{sac} + |\tau|\right) + 2BT_{sac}\frac{2}{\pi}\int_0^{\pi/2} \operatorname{sinc}\left(kv_{sac}T_{sac}\cos\theta\right) d\theta + B\left(|\tau| - T_{sac}\right) J_0\left(kv_{sac}T_{sac}\right)
\end{aligned} \quad , (6.4.13)
$$

where we have used $\frac{2}{\pi}\int_0^{\pi/2} \cos\left(u\cos\theta\right) d\theta = J_0(u)$.

Reconsider the justification for the averaging over θ. For saccades in the horizontal (x)-direction, we can write $q_{2D}(\Delta x,\Delta y,\tau;v_{sac}) = q_{1D}(\Delta x,\tau;v_{sac})\delta(\Delta y)$. Its 2D transform is $\tilde{q}_{2D}(k_x,k_y,\tau;v_{sac}) = \tilde{q}_{1D}(k_x,\tau;v_{sac})$. This is the relevant expression for all spatial frequencies, if all saccades were horizontal. In coordinate-free notation, $\tilde{q}_{2D}(\vec{k},\tau;v_{sac}) = \tilde{q}_{1D}(\left|\vec{k}\right|\cos\theta,\tau;v_{sac})$, where θ is the angle between $\vec{k}$ and the saccade velocity vector. Thus

$$\tilde{q}_{2D}(\vec{k},\tau;v_{sac}) = \left\langle \tilde{q}_{1D}(\left|\vec{k}\right|,\tau;v_{sac}\cos\theta)\right\rangle_\theta = \left\langle \tilde{q}_{1D}(\left|\vec{k}\right|\cos\theta,\tau;v_{sac})\right\rangle_\theta . \tag{6.4.14}$$

Now we need to find

$$\tilde{\tilde{q}}(k,\omega) = \int_{-\infty}^{\infty} \tilde{q}(k,\tau)\exp(-i\omega\tau)d\tau \tag{6.4.15}$$

This is done numerically for $\tilde{q}_{2D}(k,\tau)$ in eyemov_sac_demo.m.

Note that there is a problem with doing the integration in a straightforward way, since for large τ, both (6.4.11) and (6.4.13) grow in proportion to $|\tau|$, and there's a constant term in addition – making the Fourier integral diverge. Look instead at

$$\tilde{q}_{1D;loc}(k,\tau) = \tilde{q}_{1D}(k,\tau) + B\left(T_{sac}+|\tau|\right) - 2BT_{sac}\,\mathrm{sinc}\left(kv_{sac}T_{sac}\right) - B\left(|\tau|-T_{sac}\right)\cos\left(kv_{sac}T_{sac}\right), \tag{6.4.16}$$

which removes the terms that cause divergence.
For $|\tau| \le T_{sac}$,

$$\begin{aligned}\tilde{q}_{1D;loc}(k,\tau) &= 1 + B\left(T_{sac}+|\tau|\right)\mathrm{sinc}(kv_{sac}|\tau|) - 2BT_{sac}\,\mathrm{sinc}\left(kv_{sac}T_{sac}\right) - B\left(|\tau|-T_{sac}\right)\cos\left(kv_{sac}T_{sac}\right)\\ &= 1 + B\left(T_{sac}+|\tau|\right)\mathrm{sinc}(kv_{sac}|\tau|) - B|\tau|\cos\left(kv_{sac}T_{sac}\right) - 2BT_{sac}\,\mathrm{sinc}\left(kv_{sac}T_{sac}\right) + BT_{sac}\cos\left(kv_{sac}T_{sac}\right)\end{aligned}. \tag{6.4.17}$$

For $|\tau| \ge T_{sac}$,

$$\tilde{q}_{1D;loc}(k,\tau) = 1 . \tag{6.4.18}$$

The variable part of $\tilde{q}_{1D;loc}(k,\tau)$, $\tilde{q}_{1D;varloc}(k,\tau) = \frac{1}{B}\left(\tilde{q}_{1D;loc}(k,\tau)-1\right)$, is, for $|\tau| \le T_{sac}$,

$$\tilde{q}_{1D;varloc}(k,\tau) = \left(T_{sac}+|\tau|\right)\mathrm{sinc}(kv_{sac}|\tau|) - |\tau|\cos\left(kv_{sac}T_{sac}\right) - 2T_{sac}\,\mathrm{sinc}\left(kv_{sac}T_{sac}\right) + T_{sac}\cos\left(kv_{sac}T_{sac}\right). \tag{6.4.19}$$

and otherwise zero. So

$$\tilde{\tilde{q}}_{1D;varloc}(k,\omega) = \int_{-\infty}^{\infty} \tilde{q}_{1D;varloc}(k,\tau)\exp(-i\omega\tau)d\tau = \int_{-T_{sac}}^{T_{sac}} \tilde{q}_{1D;varloc}(k,\tau)\exp(-i\omega\tau)d\tau , \tag{6.4.20}$$

and each term can be done in closed form.

Limiting case, infinitely fast saccades: saccade duration is crucial.

Let $T_{sac} \to 0$ and $v_{sac} \to \infty$ so that $T_{sac}v_{sac} = L_{sac}$. Then eq. (6.4.11) holds for all τ, and becomes

$$\tilde{q}_{1D}(k,\tau) = 1 - B|\tau| + B|\tau|\cos\left(kL_{sac}\right). \tag{6.4.21}$$

Eq. (6.4.13)also holds for all τ, and becomes

$$\tilde{q}_{2D}(k,\tau) = 1 - B|\tau| + B|\tau| J_0(kL_{sac}). \qquad (6.4.22)$$

In both cases, the Fourier transform of the variable part (the multiplier of B, $\tilde{q}_{var}(k,\tau) = (\tilde{q}(k,\tau) - 1)/B$) will be a constant times the Fourier transform of $|\tau|$, i.e., the Fourier transform of the formal integral of a Heaviside function. Heaviside Fourier transform is $(i\omega)^{-1}$, so its integral So, the Fourier transform of $|\tau|$ is $(i\omega)^{-2} = -1/\omega^2$, yielding

$$\tilde{\tilde{q}}_{1D,var}(\omega,\tau) = (1 - \cos(kL_{sac}))/\omega^2 \qquad (6.4.23)$$

and

$$\tilde{\tilde{q}}_{2D,var}(\omega,\tau) = (1 - J_0(kL_{sac}))/\omega^2. \qquad (6.4.24)$$

In these cases, there is no spatiotemporal coupling. So saccade duration is crucial.

Some notes on alternative models

Model for eye movements with partial stabilization

Motivated by discussions with Michele and Xutao – one can model Kelly's system by

Residual eye position(t)=eye position(t)-[A*eye position(t-delay)+B], where A is gain of system (ideally 1 but not really), delay is about 6 ms, and B is offset, but perhaps should be B(t) for some slower-drift-rate Brownian process

Self-avoiding walk models

Roberts, Wallis, Breakspear, Sfn 2011: 379.07:random walk model for eye fixational eye movements: Parameters are simulation interval, the forgetting time for previous location, and the rewsolution of the previous location; a random step in a field with a penalty (Gaussian potential) for return to a recent location – behavior is supra-diffusive when memory of previous locations causes self-avoidance; becomes nearly Brownian (mean-squared disp prop to time) for times > 100 ms, once memory fades;
also probably becomes nearly Brownian for very short times, because of "thermal" behavior

Another SA walk: Engbert, Mergenthaler, Sinna, Pikovsy, An integrated model of fixational eye movements and microsaccades, PNAS

Also, Markovian models based on a distribution of angles w.r.t. previous step (steps defined by sampling at some short interval, e.g., 1 ms)